KMB, 14/12/17

M1F Foundations of Analysis, Problem Sheet 10 solutions

1.

(i) Choose $y \in Y$ and set $x = g(y)$. We are given $f \circ g$ is the identity function, and this implies $(f \circ g)(y) = y$, so $f(g(y)) = y$ so $f(x) = y$. Hence y is in the image of f. But $y \in Y$ was arbitrary, hence f is surjective.

(ii) Say $x_1, x_2 \in X$ and $f(x_1) = f(x_2)$. Applying g we deduce that $g(f(x_1)) = g(f(x_2))$. But $g \circ f$ is the identity function $X \to X$, so $(g \circ f)(x) = x$ for all $x \in X$. In particular $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$, and hence f is injective.

2.

(i) $(f \circ g)(x) = f(g(x)) = f(2x) = (2x)^2 + 3 = 4x^2 + 3$.
(ii) $(g \circ f)(x) = g(f(x)) = g(x^2 + 3) = 2x^2 + 6$.
(iii) The function sends x to $f(x)g(x) = (x^2 + 3)(2x) = 2x^3 + 6x$.
(iv) The function sends x to $f(x) + g(x) = x^2 + 3 + 2x = x^2 + 2x + 3$.
(v) $x \mapsto f(g(x))$ is the same function as $x \mapsto (f \circ g)(x)$ so the answer is the same as (i).

3. To check that $(f \circ g) \circ h = f \circ (g \circ h)$ we first observe that all the instances of $\circ$ actually make sense (for example $f : C \to D$ and $g : B \to C$, so $(f \circ g)$ is a well-defined function $B \to D$ etc) and as an outcome of this we see that $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are both functions $A \to D$ and in particular it makes sense to ask if they are equal.

Now what does it *mean* for two functions $A \to D$ to be equal? It means that for all $a \in A$, the values of the two functions coincide at a. In other words, in our case it means that $((f \circ g) \circ h)(a) = (f \circ (g \circ h))(a)$, so this is what we need to check. However, by continually appealing to the definition of $\circ$, we see that

$$\begin{aligned} &((f \circ g) \circ h)(a) \\ &= (f \circ g)(h(a)) \\ &= f(g(h(a))) \end{aligned}$$

and

$$\begin{aligned} &(f \circ (g \circ h)(a) \\ &= f((g \circ h)(a)) \\ &= f(g(h(a))) \end{aligned}$$

so both $((f \circ g) \circ h)(a)$ and $(f \circ (g \circ h))(a)$ equal $f(g(h(a)))$, which, let's face it, are the only possible thing that they could have equalled, because there is no other conceivable way of defining an element of D given an element of A.

4.

(i) Say $S = \{s_1, s_2, \ldots, s_n\}$. If $f : S \to T$ is an injection then the n elements $f(s_i)$ for $1 \leq i \leq n$ are distinct, so we can enumerate the elements of T as $f(s_1), f(s_2), \ldots, f(s_n), t_1, t_2, \ldots, t_r$ where r (which could be zero) is the number of elements of T which are not in the image of S. We see that $m = n + r$ and hence $m \geq n$.

(ii) Say $T = \{t_1, t_2, \ldots, t_m\}$. For $1 \leq i \leq m$ define $S_i = \{s \in S : f(s) = t_i\}$. Then the S_i are disjoint and their union is S. In particular, if σ_i denotes the size of S_i then $\sigma_1 + \sigma_2 + \cdots + \sigma_m = n$. However f is surjective and hence each S_i is non-empty, which implies that each σ_i is at least 1. Hence $n = \sum_{i=1}^{m} \sigma_i \geq \sum_{i=1}^{m} 1 = m$.

(iii) Follows immediately from (i) and (ii).

5.

(i) Let's count X: in other words let's write $X = \{x_1, x_2, x_3, \ldots\}$. Then Y is a subset of X, so Y looks like $\{x_3, x_{10}, x_{12345}, \ldots\}$ and what we need to do is to come up with a bijection between this and $\mathbf{N}$. But it's clear how to do such a thing – for the Y above we would set $f(1) = x_3$, $f(2) = x_{10}$ and so on, and in general Y must have the form $\{x_s : s \in S\}$ where S is an infinite subset of $\mathbf{N}$, and so we can write $S = \{s_1, s_2, s_3, \ldots\}$ (listing the elements in the order that they occur in our list of elements of X) and then define our bijection $\mathbf{N} \to Y$ by sending n to x_{s_n}.

(ii) If A and B are countable, then count them as $A = \{a_1, a_2, a_3, \ldots\}$ and $B = \{b_1, b_2, b_3 \ldots\}$. Now here's an injection from $A \cup B$ to $\mathbf{N}$: if $a \in A$ then $a = a_n$ for some n and we send a to $2n$ (note in particular that if $a_n \in A \cap B$ then it gets sent to $2n$). If on the other hand $x \in A \cup B$ is not in A, then it must be in B so it's of the form b_m for some m and we send x to $2m - 1$.

This is easily checked to be an injection $A \cup B \to \mathbf{N}$ and so by (i) we see $A \cup B$ is countably infinite.

(iii) If we fix bijections $f : \mathbf{N} \to A$ and $g : \mathbf{N} \to B$ with inverses f^{-1} and g^{-1}, then one checks easily that $f \times g : \mathbf{N} \times N \to A \times B$, defined by sending (i, j) to $(f(i), g(j))$ is a bijection, because it has a two-sided inverse $f^{-1} \times g^{-1}$ sending (a, b) to $(f^{-1}(a), g^{-1}(b))$. In short, it suffices to prove that $\mathbf{N} \times \mathbf{N}$ is countable. One can count it directly via a "wiggly line" construction like in lectures, but here's another cute argument: consider $h : \mathbf{N}^2 \to \mathbf{N}$ defined by $h(i, j) = 2^i 3^j$. By uniqueness of prime factorization, h is injective, so the result follows from (i).

(iv) The reals are not countable but the rationals are (proved in lectures), so if the irrationals were also countable then then the reals would be too by (ii), a contradiction. Thus the irrationals must be uncountable. The complexes are clearly uncountable because they contain the reals which are uncountable so we're done by (i). Finally $\mathbf{Q}(i)$ is countable, because as a set it clearly bijects with $\mathbf{Q} \times \mathbf{Q}$, and the product of two countable sets is countable by (iii).

6.

(i) The right hand size is $\frac{n!}{r!(n-r)!} + \frac{n!}{(r-1)!(n-r+1)!}$ and putting this over a common denominator gives $\frac{n!(n-r+1)+n!(r)}{r!(n-r+1)!} = \frac{n!(n+1)}{r!(n-r+1)!} = \binom{n+1}{r}$.

For a proof not relying on the formulae, one can note that the left hand side expresses the number of ways we can choose r things from $(n+1)$ things. Here's one way of counting the number of ways: imagine one of the $n+1$ things is called Kevin. When we choose r things we either choose Kevin or we don't. If we choose Kevin then to finish the job we need to choose $r-1$ of the remaining n things, and we can do this in $\binom{n}{r-1}$ ways. However if we don't then we need to choose r things from the remaining n, which we can do in $\binom{n}{r}$ ways. Hence $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

(ii) The cases $n = 0, 1$ are easy. If it's true for $n = d$ then

$$\begin{aligned}
(x+y)^{d+1} &= (x+y)(x+y)^d \\
&= (x+y)\left(\binom{d}{0}x^d + \binom{d}{1}x^{d-1}y + \binom{d}{2}x^{d-2}y^2 + \cdots + \binom{d}{d}y^d\right) \\
&= \binom{d}{0}x^{d+1} + \left(\binom{d}{0} + \binom{d}{1}\right)x^d y + \left(\binom{d}{1} + \binom{d}{2}\right)x^{d-1}y^2 + \cdots + \left(\binom{d}{d-1} + \binom{d}{d}\right)xy^d + \binom{d}{d}y^{d+1} \\
&= x^{d+1} + \binom{d+1}{1}x^d y + \cdots + \binom{d+1}{d}xy^d + y^{d+1} \\
&= \sum_i \binom{d+1}{i} x^{d+1-i}y^i
\end{aligned}$$

which is what we wanted.

7. (i) For a prime p to divide the product of finitely many numbers it must divide at least one of them – this is Proposition 6.6 from the course. In particular at least one of the numbers must be at least p. However if $1 \le i \le p-1$ then none of the integers whose product is $i!$ or $(p-i)!$ are at least p, so p cannot divide $i!(p-i)!$.

We know $\binom{p}{i} i!(p-i)! = p!$, and the right hand side is clearly a multiple of p, and p does not divide $i!(p-i)!$ so by Proposition 6.5 we must have $p \mid \binom{p}{i}$.

(ii) The statement is clearly true for $a = 0$ (and $a = 1$). If it's true for $a = d$ then $(d+1)^p = d^p + \binom{p}{1}d^{p-1} + \binom{p}{2}d^{p-2} + \cdots + \binom{p}{p-1}d + 1$. All the terms in the middle are congruent to 0 mod p

by (i), and hence $(d+1)^p \equiv d^p + 1 \bmod p$. The inductive hypothesis says $d^p \equiv d \bmod p$, and we are now done.

8. The answer is the number of ways we can break up our 9 slots for letters into subsets of size 1 (E), 3 (M), 1 (A), 2 (C), 1 (O) and 1 (Y), so by definition the answer is $\binom{9}{1,3,1,2,1,1}$ which is $9!/(1!3!1!2!1!1!) = 9!/12 = 30240$.

9. (i) A direct application of the binomial theorem tells us that the answer is $\binom{21}{19} = \frac{20\times 21}{2} = 210$.

(ii) The question asks us to find the coefficient of x in $\left(x^{-1}(x^4+1)\right)^7 = x^{-7}(x^4+1)^7$, and this is the same as the coefficient of $x^{1+7} = x^8$ in $(x^4+1)^7$, so setting $y = x^4$ we're asking for the coefficient of y^2 in $(y+1)^7$, which is $\binom{7}{2} = \frac{7\times 6}{2} = 21$.

(iii) The general term in the expansion of $(1+3x+x^2)^5$ is $\binom{5}{a,b,c} 1^a (3x)^b x^{2c}$, with $a, b, c \geq 0$ and $a+b+c = 5$. Now the degree of $1^a(3x)^b x^{2c}$ is $b+2c$, so for the terms we're interested in we must have $a+b+c = 5$ and $b+2c = 6$, hence (subtracting) $c - a = 1$. Because $c + a \leq 5$ we can only have $(a,c) = (0,1), (1,2)$ or $(2,3)$ giving $(a,b,c) = (0,4,1), (1,2,2), (2,0,3)$. These are the (a,b,c) values corresponding to the terms in the expansion which have degree 6. So the answer to the question is $\binom{5}{0,4,1}3^4 + \binom{5}{1,2,2}3^2 + \binom{5}{2,0,3}3^0 = 5\times 3^4 + 30\times 3^2 + 10 = 685$.