KMB, 3/10/17

M1F Foundations of Analysis, Problem Sheet 1 solutions.

1.

(a) F ($x = 2$ is also a root)
(b) T (it doesn't matter that $x = 2$ is a root here)
(c) F ($x = 2$ is a problem again)
(d) T (the two roots are $x = 1$ and $x = 2$ – can you prove there are no others?)
(e) T ($x = 3$ isn't a root but this doesn't matter)
(f) F ($x = 3$ isn't a root and this time it matters).

The key thing to understand here is that $P \Rightarrow Q$ means, and *only* means, that if P is true, then Q is true. So, for example, part (e) is true, even though in practice it's a bit weird and unhelpful; the point is that logically it's a true statement.

2. It is true that R implies P. Here's why. Let's assume R is true (with the goal of trying to deduce that P is true). Can Q be false? No! For if Q is false then we know from the question that R will also be false, but we're assuming R is true. So Q must be true as well. And then again from the question, P must be true. So if R is true then P is true too.

3. P is true and Q is false, so $(P \Rightarrow Q)$ is false. Similarly, R is false and S is true, so $(R \Rightarrow S)$ is true. So the question asks whether (false) $\Leftarrow$ (true), and this is false. So the answer to the question is "false".

4. Let's consider the possibilities for P.

(i) Say P is true. Then (a) tells us that either Q or R (or both) are true. However (c) tells us that if Q *and* R are true, then $\neg P$ is true, which would mean that P is false, a contradiction. We conclude that if P is true then *exactly one* of Q and R are true.

We've not thought about (b) yet so let's do that now. The right hand side of (b) is $R \vee \neg P$ which (because we're assuming R is true) is the same as $R \vee$ (false), which is the same as R. So (b) just says $\neg Q \Rightarrow R$. We know that exactly one of Q and R is true, and the other is false. So (b) either says (false) $\Rightarrow$ (false) or (true) $\Rightarrow$ (true), and both of these are indeed true.

Conclusion: if P is true, then either Q is true and R is false, or Q is false and R is true.

(ii) Now let's say P is false. Then (a) is automatically true whatever Q and R are (because false implies anything), and (b) and (c) are also automatically true (because $R \vee \neg P$ and $\neg P$ are both true, and anything implies true).

Conclusion: If P is false, Q and R can be arbitrary.

Overall conclusion: we can make no individual deductions about any of P, Q or R. The complete list of possibilities for PQR is $TTF, TFT, FTT, FTF, FFT, FFF$.

5. TFTTTFFT.

The only tricky thing here is to understand that $\{1, 2, 1\} = \{1, 2\}$.

6. TFTTTFTT.

Note: $\{1, 2\} \in A$ and $A \in B$, but $\{1, 2\} \notin B$.

7.

(a) is false, because $B \subseteq \mathbf{Z}$ by definition, so $\frac{1}{2} \notin B$, so $\frac{1}{2} \notin A \cap B$.

(b) is true, because $\frac{1}{2} \in \mathbf{R}$ and $\left(\frac{1}{2}\right)^2 = \frac{1}{4} < 3$, so $\frac{1}{2} \in A$ and hence $\frac{1}{2} \in A \cup B$.

(c) is false, because $x = \frac{3}{2} \in \mathbf{R}$ satisfies $x^2 = 9/4 < 3$ and $x^3 = 27/8 > 3$, so $x \in A$ but $x \notin C$, which means $A \not\subseteq C$.

(d) is true. In fact if $x \in \mathbf{Z}$ and $|x| \geq 2$ then $x^2 \geq 4$, meaning $x \notin B$. On the other hand ± 1 and $0 \in B$, meaning $B = \{-1, 0, 1\}$. All of these elements are easily checked to be in C.

(e) is not true, because $-100 \in C$ (as its cube is less than zero) but $(-100)^2 = 10^4 > 3$ so $-100 \notin A$ and $-100 \notin B$.

(f) is not true. To check that these sets are not equal, all we need to do is to find a real number which is in one but not the other. Again I claim $x = -100$ will work. For we've just seen it's in C, so it's definitely in $(A \cap B) \cup C$. However we've also just seen it's not in $A \cup B$, so it's also not in $(A \cup B) \cap C$.