KMB, 18/11/17

M1F Foundations of Analysis, Problem Sheet 6.

1. A *regular polygon* is a 2-dimensional shape (so one 2-d face) with all edges of equal length and all internal angles between adjacent edges equal. Examples: an equilateral triangle, a square, a regular pentagon etc.

A *regular polyhedron* is a 3-dimensional shape, which is a convex polyhedron such that all of its faces are copies (all the same size) of one fixed regular polygon (e.g. they could be all squares, or all triangles), and *furthermore* such that every vertex has the same number of faces meeting at it. Examples: the regular tetrahedron, or the cube (both of which have three faces meeting at each vertex).

a) To help you understand the concept of a regular polyhedron, let me give you an example of a polyhedron made up of equilateral triangles which is *not* regular. So take two regular tetrahedra and then glue a face of one onto a face of the other. The resulting polyhedron now has 6 faces. Count the number of vertices and the number of edges, and check that $V - E + F = 2$ is true. Why is this polyhedron *not* regular?

b) Now say X is a regular polyhedron, with F faces each of which have n sides (i.e., n edges per face), E edges, and V vertices each of which is where r faces meet (note we must have $r \geq 3$ for our polyhedron to have non-zero volume!). By counting edges in three ways (looking at faces, edges and vertices), prove that $nF = 2E = rV$.

c) Let's contemplate the existence of a regular polyhedron made up of pentagons. With notation as above we then have $n = 5$. Because the interior angle of a regular pentagon is 108 degrees, if such a polyhedron existed it must have $r = 3$ Using the formula from (b) (as $r = 4$ gives us more than 360 degrees). Now use (b) and $V - E + F = 2$ to deduce how many vertices, edges and faces such a shape must have.

d) Does the calculation in (c) prove the existence of the dodecahedron?

2. Say G is a (finite) connected planar graph with v vertices, e edges and f faces, and each face has at least three sides (this would be the case if, for example, all the edges of our graph were straight lines).

a) By counting faces, show $3f \leq 2e$.

b) Deduce that there must be a vertex in G with at most 5 edges coming from it.

c) Can you find an infinite connected planar graph with straight lines for edges and with each vertex having 6 edges coming from it?

3*. For each of the following non-empty sets S, figure out whether or not they are bounded above. For those that are bounded above, figure out what the least upper bound is. Full proofs required!

a) $S = (-\infty, 0)$

b) $S = \mathbf{Q}$

c) $S = \{x \in \mathbf{R} : (x+1)^2 < x^2\}$

d) $S = \{x \in \mathbf{Q} : 1 < x < 2\}$

4. Say $S \subset \mathbf{R}$, and S has an upper bound $x \in \mathbf{R}$ with the property that $x \in S$. Prove that x is the least upper bound for S.

5. If S is a set of real numbers, we say S is *bounded below* if there exists some $x \in \mathbf{R}$ with $x \leq s$ for all $s \in S$; such an x is called a *lower bound* for S; we say $z \in \mathbf{R}$ is a *greatest lower bound* (GLB) for S if z is a lower bound for S and furthermore that if $y \in \mathbf{R}$ is any lower bound then $z \geq y$.

a) Prove that S is bounded below if and only if $-S := \{-s : s \in S\}$ is bounded above. Then prove that x is a greatest lower bound for S if and only if $-x$ is a least upper bound for $-S$.

b) Prove that if x_1 and x_2 are both greatest lower bounds for S, then $x_1 = x_2$.

c) Assuming that any non-empty bounded-above set of reals has a LUB, prove that any non-empty bounded-below set of reals has a GLB.

6. This question is quite fun.

Say we have a sequence of real numbers a_1, a_2, $a_3, \ldots$, which is bounded above in the sense that there exists some real number B such that $a_i \leq B$ for all i.

Now let's define some sets S_1, S_2, $S_3, \ldots$ by

$$S_n = \{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

For example $S_3 = \{a_3, a_4, a_5, \ldots\}$.

a) Prove that for all $n \geq 1$, S_n is a non-empty set which is bounded above, and hence has a least upper bound b_n.

b) Prove that $b_{n+1} \leq b_n$ and hence b_1, b_2, b_3 is a decreasing sequence.

If the set $\{b_1, b_2, b_3, \ldots\}$ is bounded *below*, then its greatest lower bound ℓ is called the *limsup* of the sequence $(a_1, a_2, a_3, \ldots)$ (this is an abbreviation for Limit Superior).

c) Find the limsup of the following sequences (they do exist).

i) $1, 1, 1, 1, 1, \ldots$

ii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$

iii) $0, 1, 0, 1, 0, 1, 0, 1, \ldots$

d) If you like, then guess the definition of *liminf* (Limit Inferior) and compute it for examples (i) to (iii) of (c) above. Which of these sequences converges? Can you tell just from looking at the limsup and liminf?