KMB, 15th Dec 2014.

M2PM2 Algebra II, Solutions to Problem Sheet 9.

1. $P = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ (many other P's work).

2. As the only eigenvalue is 0, the char poly must be x^n. So by Cayley–Hamilton, $A^n = 0$.

3. By induction on n. The char poly is

$$p(x) = det \begin{pmatrix} x & 0 & 0 & \cdots & 0 & a_0 \\ -1 & x & 0 & \cdots & 0 & a_1 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & -1 & x + a_{n-1} \end{pmatrix}$$

Expand along the first row. By induction the det of the (1,1)-minor is $x^{n-1} + a_{n-1}x^{n-2} + \cdots + a_1$, and the $(1, n)$-minor is upper-triangular so has determinant $(-1)^{n-1}$. We deduce

$$p(x) = x\,(x^{n-1}+a_{n-1}x^{n-2}+\cdots+a_1)+(-1)^{n-1}a_0.(-1)^{n-1} = x^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0.$$

Hence the result by induction.

4. (a) $\begin{pmatrix} 0 & 0 & 3 \\ 1 & 0 & -2 \\ 0 & 1 & 7 \end{pmatrix}$ works (by Q3).

(b) If we find A with char poly $x^3 - 2x^2 - 1$ then A will satisfy the desired equation by Cayley–Hamilton. So take $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 2 \end{pmatrix}$.

(c) Multiplying through by B, the eqn is $B^4 + B - I = 0$. So finding B with char poly $x^4 + x - 1$ will do. Use Q3 to do this.

(d) By Q3 the 2×2 matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ satisfies $A^2 + A + I = 0$. So take $C = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$.

(e) Use Q3 to get a non-identity $n \times n$ matrix with char poly $x^n - 1$.

5. (i) Yes: if B is similar to A then $B^3 - I$ is similar to $A^3 - I$, so $\text{rank}(B^3 - I) = \text{rank}(A^3 - I)$ (because they are both the rank of the same linear map).

(ii) Yes: same proof shows $A + A^5$ and $B + B^5$ are similar, so it suffices to check that similar matrices have the same trace. But the trace is (up to sign) one of the coefficients of the char poly!

(iii) No: eg $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ are similar but have different first column sum.

(iv) No: eg let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 2 \\ 1/2 & 0 \end{pmatrix}$. Then A and B are similar, but $A - A^T = 0$ has rank 0, whereas $B - B^T$ has rank 2.

(v) Yes: A and A^T have the same diagonal entries, so $\text{trace}(2A - A^T) = \text{trace}(A)$, which is invariant as we saw in part (ii).

6. This question is fairly easy, but notationally awkward. Say each A_i is $n_i \times n_i$, so A is $n \times n$ where $n = \sum n_i$. Write each column vector in F^n ($F = \mathbb{R}$ or $\mathbb{C}$) in the form $v = (v_1, v_2, \dots, v_k)$, where $v_i \in F^{n_i}$ for all i. Then $Av = (A_1v_1, A_2v_2, \dots, A_kv_k)$. Hence $Av = \lambda v$ if and only if $A_iv_i = \lambda v_i$ for all i.

Let $E_\lambda(A_i)$ be the λ-eigenspace of A_i, and let B_i be a basis of $E_\lambda(A_i)$. Each vector $b \in B_i$ gives a vector $(0, \dots, b, \dots 0)$ in F^n. Let B_i' be the set of such vectors obtained from B_i. By the previous observation, vectors in $E_\lambda(A)$ are of the form $(v_1, v_2, \dots, v_k)$ with $v_i \in E_\lambda(A_i)$. These are linear combinations of the vectors in $\cup B_i'$. Hence $\cup B_i'$ is a basis for $E_\lambda(A)$. So $\dim E_\lambda(A) = \sum |B_i'| = \sum |B_i| = \sum \dim E_\lambda(A_i)$.

7. (i) There is one possibility for the 0-blocks, two for the $-1-i$-blocks and three for the 3-blocks, giving a total of $1 \times 2 \times 3 = 6$ possibilities. In full, they are $J_1(0) \oplus J_1(-1-i)^{\oplus 2} \oplus J_1(3)^{\oplus 3}$, $J_1(0) \oplus J_1(-1-i)^{\oplus 2} \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_1(-1-i)^{\oplus 2} \oplus J_3(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_1(3)^3$, $J_1(0) \oplus J_2(-1-i) \oplus J_2(3) \oplus J_1(3)$, $J_1(0) \oplus J_2(-1-i) \oplus J_3(3)$.

(ii) There are 3 possible JCFs with char poly x^3 ($J_3(0)$, $J_2(0) \oplus J_1(0)$ etc) and 11 with char poly $(x-1)^6$ ($J_6(1)$, $J_5(1) \oplus J_1(1)$ etc). So there are 33 JCFs with char poly $x^3(x-1)^6$.

8. In the proof of uniqueness of decomposition into Jordan blocks we saw that the sizes of the blocks can be read off from the ranks of $(A - \lambda I)^j$ for $j = 1, 2, 3, 4, \dots$. Applying the technique in this proof gives:

$J_1(1) \oplus J_1(0) \oplus J_1(-1)$, $J_1(3) \oplus J_1(0)^{\oplus 2}$, $J_1(-1) \oplus J_2(2)$, $J_4(0) \oplus J_1(0)$, $J_3(-1) \oplus J_1(-1) \oplus J_2(i)$.

9. Let E be the standard basis in order $e_1, \dots, e_n$ and F the standard basis in reverse order $e_n \dots, e_1$. As $Je_n = e_{n-1}$, $Je_{n-1} = e_{n-2}$, etc, the linear transformation $T(v) = Jv$ satisfies $[T]_E = J$, $[T]_F = J^T$. So if P is the change of basis matrix from E to F, $P^{-1}JP = J^T$. Therefore J and J^T are similar.

Finally,

$$P^{-1}J_n(\lambda)P = P^{-1}(J + \lambda I)P = J^T + \lambda I = (J + \lambda I)^T = J_n(\lambda)^T$$

so $J_n(\lambda)$ and $J_n(\lambda)^T$ are similar.

10. Let A be an $n \times n$ matrix over $\mathbb{C}$. By the JCF theorem A is similar to a JCF matrix $J = J_{n_1}(\lambda_1) \oplus \cdots \oplus J_{n_k}(\lambda_k)$. By Q9, for each i, $\exists P_i$ such that $P_i^{-1}J_{n_i}(\lambda_i)P_i = J_{n_i}(\lambda_i)^T$. If we let P be the block-diagonal matrix $P_1 \oplus \cdots \oplus P_k$, then $P^{-1} = P_1^{-1} \oplus \cdots \oplus P_k^{-1}$ and so

$$P^{-1}JP = P_1^{-1}J_{n_1}(\lambda_1)P_1 \oplus \cdots \oplus P_k^{-1}J_{n_k}(\lambda_k)P_k = J_{n_1}(\lambda_1)^T \oplus \cdots \oplus J_{n_k}(\lambda_k)^T = J^T.$$

So J is similar to J^T, and hence A is similar to J^T, i.e. $\exists Q$ such that $Q^{-1}AQ = J^T$.

Taking transposes, $Q^T A^T (Q^{-1})^T = J$. Since $(Q^{-1})^T = (Q^T)^{-1}$, this shows A^T is similar to J. So both A and A^T are similar to J, whence A is similar to A^T.

11. (i) E.g. $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

(ii) This is a really nice application of the JCF (and I'm not sure I know a more satisfactory way to do the question). Here's a sketch of the argument. Since A is similar to a block diagonal sum of Jordan blocks $J_r(\lambda)$ (with $\lambda \neq 0$ as A is invertible), it is enough to show that each such Jordan block $J_r(\lambda)$ has a square root. Let μ be a square root of λ in $\mathbb{C}$, so $\mu \neq 0$. Consider $J_r(\mu) = J + \mu I$ where

$$J = \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ & & & \cdots \end{pmatrix}.$$

Then $J_r(\mu)^2 = J^2 + 2\mu J + \mu^2 I$. In particular $J_r(\mu)^2$ is upper-triangular and its only eigenvalue is μ^2. Next, check by looking at row or column ranks that the rank of $J_r(\mu)^2 - \mu^2 I$ is $r - 1$ (this is where we assume $\mu \neq 0$, and for you boffins doing the entire course over an arbitrary abstract field it's also the place where we assume $2 \neq 0$) and hence the nullity is 1, so $J_r(\mu)^2$ has eigenvalue μ^2 with algebraic multiplicity r and geometric multiplicity 1. Hence its JCF must be $J_r(\mu^2) = J_r(\lambda)$. Hence $\exists P$ such that $P^{-1} J_r(\mu)^2 P = J_r(\lambda)$, i.e. $(P^{-1} J_r(\mu) P)^2 = J_r(\lambda)$. Hence $J_r(\lambda)$ has a square root, as desired.