KMB, 1st December 2014

M2PM2 Algebra II: Solutions to Problem Sheet 8

1. Define $A \sim B$ if $\exists P$ invertible such that $B = P^{-1}AP$.

Then $A \sim A$ as $A = I^{-1}AI$.

And $A \sim B \Rightarrow B = P^{-1}AP \Rightarrow A = PBP^{-1} \Rightarrow B \sim A$.

Finally $A \sim B$, $B \sim C \Rightarrow B = P^{-1}AP$, $C = Q^{-1}BQ \Rightarrow C = Q^{-1}P^{-1}APQ = (PQ)^{-1}A(PQ) \Rightarrow A \sim C$.

Hence $\sim$ is an equivalence relation.

2. Routine first year stuff: $P = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$, $Q = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}$, $[v]_E = (a,b)^T$, $[v]_F = (-5a+2b, 3a-b)^T$, $[T]_E = \begin{pmatrix} 0 & 2 \\ 3 & -1 \end{pmatrix}$, $[T]_F = \begin{pmatrix} -30 & -48 \\ 18 & 29 \end{pmatrix}$.

3. (i) The determinant must be 0 because T is not surjective. More precisely, if $T(x_1,x_2,x_3) = (y_1,y_2,y_3)$ with y_i defined as in the question, then one spots $y_1+y_2+y_3 = (x_1-x_2+2x_3)+(-x_1-3x_3)+(x_2+x_3) = 0$. This means that the image of T is contained in the 2-dimensional subspace of $\mathbb{R}^3$ consisting of vectors whose entries sum to zero. Because the image of T cannot be 3-dimensional, the Rank-Nullity theorem tells us that the kernel cannot be 0-dimensional. So there are vectors in the kernel of T, which means that T cannot be invertible and the determinant must then be zero.

Alternatively, just bash it out. Write down the matrix representing T with respect to the obvious basis and then observe that the three rows of the resulting matrix sum to zero, so the determinant must be zero.

(ii) The matrix of T w.r.t the usual basis $1, x, x^2, x^3$ is triangular with diagonal entries all 1, so has determinant 1.

(iii) Matrix of T w.r.t. basis $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is $A = \begin{pmatrix} 1 & -2 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix}$, which has determinant equal to $(\det(M))^2 = 36$.

4. The T of Q3(ii) satisfies $T(1) = 1$, $T(x) = x$, $T(x^2) = x^2+4x-1$, and $T(x^3) = x^3+9x-2$, so matrix of T w.r.t. basis $1, x, x^2, x^3$ is

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & 4 & 9 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The only eigenvalue is 1, and basis for 1-eigenspace of T is checked to be $1, x$ without too much trouble. There is hence no basis of evectors: $g(1) < a(1)$.

The T of Q3(iii) has matrix A as in the solution above. The characteristic polynomial of the matrix $\begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$ is $(x-2)(x-3)$ so this matrix has distinct evalues so can be diagonalised, say by a 2×2 matrix P. Then the 4×4 matrix $\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix}$ diagonalises A. The eigenspaces are $\begin{pmatrix} -2a & -2b \\ a & b \end{pmatrix}$ and $\begin{pmatrix} a & b \\ -a & -b \end{pmatrix}$.

5. (a)(i) Characteristic polynomial is $(x+1)^2(x-2)$, so eigenvalues are $-1, 2$ with algebraic multiplicities 2,1 respectively. The geometric multiplicity of the evalue -1 is dimension of the -1 eigenspace, which is easily checked to be 1; the geometric multiplicity of 2 must also be 1 (as $1 \leq g(2) \leq 1$). Since the geom multiplicity of -1 is less than the algebraic multiplicity, there is no basis of eigenvectors.

(ii) T sends $1 \to 0$, $x \to 3x$, $x^2 \to x + 6x^2$, so matrix of T wrt basis $1, x, x^2$ is $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 6 \end{pmatrix}$. This has distinct eigenvalues 0,3,6, all with algebraic and geometric multiplicity 1, and there is a basis of eigenvectors.

(b) The char poly is $(x+1)^2(x-1)$, so (as in part (a)(i) above) A is diagonalisable iff the -1 eigenspace has dimension 2. This eigenspace consists of solutions to the system $\begin{pmatrix} 0 & a & b \\ 0 & 2 & c \\ 0 & 0 & 0 \end{pmatrix} v = 0$, so it is 2-dimensional iff $ac - 2b = 0$.

6. (i) Well-definedness of multiplication (i.e. "closure"): $S, T \in GL(V)$ implies that ST is a linear transformation, and it is invertible as $(ST)^{-1} = T^{-1}S^{-1}$, so $ST \in GL(V)$.

Associativity: follows from associativity of composition.

Identity: is identity map $I(v) = v \forall v \in V$.

Inverse: exists by defition.

Hence $GL(V)$ is a group.

(ii) If $T, U \in GL(V)$ then $\det(TU) = \det(T)\det(U)$ by lectures, so det is a homomorphism. Let $B = \{v_1, \ldots, v_n\}$ be a basis of V. For $\lambda \in \mathbb{R}^*$, define $T : V \to V$ to be the linear map which sends

$$v_1 \to \lambda v_1, v_2 \to v_2, \ldots, v_n \to v_n.$$

Then $\det(T) = \lambda$. Hence det is surjective.

(iii) Fix a basis B of V. Then the map $T \to [T]_B$ is an isomorphism from $GL(V)$ to $GL(n, \mathbb{R})$.