KMB, 6/11/14

M2PM2 Algebra II: Solutions to Sheet 3.

1. (i) We saw in lectures (Prop 3.2) that an n-cycle can be written as a product of $n-1$ 2-cycles, so the answer is yes, and from the proof we can even see in this case that $g = (15)(14)(13)(12)$.

(ii) From (i) we know that $\mathrm{sgn}(g) = (-1)^4 = +1$. However a product of 2011 2-cycles will have signature $(-1)^{2013} = -1$, and hence g cannot be written as a product of 2013 2-cycles.

(iii) A product of two 2-cycles can move at most four elements of the set $\{1, 2, 3, 4, 5\}$, but g moves all five elements, and hence g cannot be a product of two 2-cycles.

2. By Proposition 5.3, $C_1 \times C_{24} = C_{24} = C_3 \times C_8$. Similarly $C_2 \times C_{12} = C_2 \times C_3 \times C_4 = C_4 \times C_2 \times C_3 = C_4 \times C_6$ (using things like Q6, which is not hard). So it just remains to show that C_{24} is not isomorphic to $C_4 \times C_6$ (because then by transitivity of $\cong$, no other pair of groups on our list will be isomorphic). However C_{24} only contains one element of order 2, and $C_4 \times C_6$ contains three (I did this in lectures for C_4 and $C_2 \times C_2$, and the arguments here are pretty much the same) and so they are not isomorphic, and everything now follows.

3. (a) There are lots of examples. For example let's consider the group $C_3 \times C_3 \times \cdots \times C_3$, with n terms in the product. An element g of this group which is not the identity element must have order 3, because if $g = (x_1, x_2, \ldots, x_n)$ with, say, x_i not the identity, then g and g^2 can't be the identity (as x_i and x_i^2 aren't), but $g^3 = (x_1^3, x_2^3, \ldots) = (e, e, e, \ldots, e)$ is the identity. Hence every element other than the identity has order 3, and so the product has $3^n - 1$ elements of order 3. So all we need to do is to choose some n such that $3^n - 1 \geq 100$, e.g. $n = 10$.

Note: there are many other examples, not of this form.

(b) If G is any correct answer to part (a), then $G \times S_3$ will be a correct answer to part (b), because the order of $(g, e) \in G \times S_3$ is just equal to the order of $g \in G$, so there are at least 100 elements of order 3, but conversely $(e, (1\ 2))$ and $(e, (1\ 3))$ do not commute. Alternatively, any sufficiently large symmetric group will also work for (b) – again there are many different examples.

4.

(a) An element of A_5 is of course also an element of S_5. An element of S_5 is a product of disjoint cycles, and if this element has order 3 then the lowest common multiple of the cycle lengths must be 3, and hence each cycle occurring must be (a 1-cycle or) a 3-cycle. But there is no space in S_5 for two disjoint 3-cycles, as $3 + 3 = 6 > 5$, and hence an element of order 3 must be a 3-cycle. Conversely any 3-cycle in S_5 has signature $(-1)^{3-1} = 1$ by Prop 4.5, and is hence in A_5. So we just need to count the number of 3-cycles in S_5. Each 3-cycle permutes three out of the five elements of $\{1, 2, 3, 4, 5\}$ and there are $\binom{5}{3} = 10$ ways to choose these three elements, but then they can be permuted in two different directions, giving an answer of 20.

(b) As in (a), an element of order 3 in A_6 must be a product of 3-cycles, and hence this time is either a 3-cycle or a product of two disjoint 3-cycles. The signatures of both of these are $+1$ so all of these are in A_6. So again we are reduced to combinatorics. Arguing as in part (a), the number of 3-cycles is $2\binom{6}{3} = 40$. The number of elements of the form $(a\ b\ c)(d\ e\ f)$ is a little tougher to count. There are $\binom{6}{3} = 20$ ways to choose a, b and c. Now there are two ways to permute them around, and two ways to permute d, e and f around, which gives a total of 80. However, we have unfortunately counted each element twice, once when we chose a, b and c, and once where we chose d, e and f. Hence the actual number is $80/2 = 40$. Hence the answer to the question is $40 + 40 = 80$.

(c) If we have an element of a symmetric group of order 8, then it is a product of disjoint cycles of lengths $r_1, r_2, \ldots, r_n$ and the lowest common multiple of the r_i must be 8. Hence (think about it) each r_i must divide 8, and one of them must *be* 8. The simplest example of course is an 8-cycle in S_8, but unfortunately this has signature $(-1)^7 = -1$ and hence is not in A_8, so we need to have at least one 8-cycle and at least one other r-cycle, for $r \geq 2$ and r dividing 8, meaning that

$n = 10$ is the smallest value of n that can possibly work, and indeed it does work, as it contains the product of an 8-cycle and a 2-cycle.

5. (a) There are many examples, eg (1623)(495)(78) and (1695)(23)(478).

(b) f is a product of n (disjoint – not that it matters) 2-cycles so has signature $(-1)^n$.

6. (a) Show $(g_1, g_2) \mapsto (g_2, g_1)$ is an isomorphism $G_1 \times G_2 \to G_2 \times G_1$.

(b) For $i = 1, 2$ let $\phi_i : G_i \to H_i$ be an isomorphism. Show that $(g_1, g_2) \mapsto (\phi_1(g_1), \phi_2(g_2))$ is an isomorphism $G_1 \times G_2 \to H_1 \times H_2$.

7. The Structure Theorem gives a complete list – the tricky part is figuring out when two groups in the list are isomorphic.

(a) One: C_{30}. (Point is that $C_2 \times C_3 \times C_5 \cong C_6 \times C_5 \cong C_2 \times C_{15} \cong C_3 \times C_{10} \cong C_{30}$ by result in lectures that $C_m \times C_n \cong C_{mn}$ if m, n are coprime.)

(b) One: C_{31}, as 31 is prime.

(c) Seven: C_{32}, $C_{16} \times C_2$, $C_8 \times C_4$, $C_8 \times (C_2)^2$, $(C_4)^2 \times C_2$, $C_4 \times (C_2)^3$ and $(C_2)^5$. Need to check no two of these are isomorphic by showing they have different numbers of elements of some order. Counting elements of order 2 shows that the only possible isomorphisms that could exist are between $C_{16} \times C_2$ and $C_8 \times C_4$, and between $C_8 \times (C_2)^2$ and $(C_4)^2 \times C_2$, but the first pair of groups (which both have three elements of order 2) aren't isomorphic because one has an element of order 16 and the other doesn't, and the second pair (which both have seven elements of order 2) aren't isomorphic either because one has an element of order 8 and the other doesn't.

8. (i) $C_2 \times \cdots \times C_2$ (n factors)

(ii) $D_8 \times D_8$, or $D_8 \times C_2$, or all sorts of other possibilities.

(iii) $\mathbb{Z} \times D_6$, where $\mathbb{Z}$ is the integers under addition. The abelian subgroup H is $\mathbb{Z} \times \langle \rho \rangle$, where ρ is a rotation of order 3 in D_6. Again, there are many other possibilities.