Imperial College London
M2PM2 Algebra II, Progress Test 3, 7/12/2012, solutions.

Q1.

i. The *determinant* of $A = (a_{ij})$ is $\sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$. One mark.

ii. If B is the matrix obtained from A by switching columns s and t, and if τ is the transposition $(s\ t) \in S_n$, then one checks easily that $b_{ij} = a_{i\tau(j)}$ for $1 \leq i, j \leq n$. Hence

$$\begin{aligned}\det(B) &= \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)b_{1\sigma(1)}b_{2\sigma(2)}\cdots \\ &= \sum_{\sigma} \operatorname{sgn}(\sigma)a_{1\tau\sigma(1)}a_{2\tau\sigma(2)}\cdots \\ &= \sum_{\pi} \operatorname{sgn}(\tau\pi)a_{1\pi(1)}a_{2\pi(2)}\cdots\end{aligned}$$

(with $\pi = \tau\sigma = \tau^{-1}\sigma$) and because $\operatorname{sgn}(\tau\pi) = -\operatorname{sgn}(\pi)$ the result follows. Two marks.

iii. $\det(C) = \sum_{\pi\in S_n} \operatorname{sgn}(\pi)c_{1\pi(1)}c_{2\pi(2)}\cdots$. If $\pi \in S_n$ and $\pi \neq \sigma$ then there exists some i such that $\pi(i) \neq \sigma(i)$ and hence $c_{i\pi(i)} = 0$, meaning that the term in the sum corresponding to π is zero. The only possible non-zero term then in the sum is the term for $\pi = \sigma$, giving

$$\det(C) = \operatorname{sgn}(\sigma)c_{1\sigma(1)}c_{2\sigma(2)}\cdots = \operatorname{sgn}(\sigma)$$

as required. Two marks.

iv. If the matrix is A, and if v is the column vector $v = (1\ \ -2\ 1)^T$, then $Av = 0$, so $\det(A)$ must be zero. One mark. Of course there are plenty of other ways of getting this mark – for example you can do some row operations, or explicitly compute the determinant.

v. For $m \geq 3$, let E_{m-1} be the $(1, 2)$ minor of D_m. Note that the first column of E_{m-1} just has one non-zero entry, namely the top left hand corner, so expanding down the first column we see $\det(E_{m-1}) = \det(D_{m-2})$.

Now say $n \geq 1$. Expanding $\det(D_{n+2})$ along the first row we see that $\det(D_{n+2}) = \det(D_{n+1}) - \det(E_{n+1})$, hence (setting $m = n + 2$) $\det(D_{n+2}) = \det(D_{n+1}) - \det(D_n)$, as required. Two marks.

vi. We check by hand that $\delta_1 = 1$ and $\delta_2 = 0$. Hence (using the recurrence proved in part (v)) $\delta_3 = -1$, $\delta_4 = -1$, $\delta_5 = 0$ and $\delta_6 = 1$. Hence D_5 is not invertible but D_6 is, because we know a square matrix is invertible iff it has non-zero determinant. Two marks.

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Q2.

i. An eigenvector with eigenvalue λ is a non-zero $v \in V$ such that $Tv = \lambda v$. One mark, but zero marks if you forget the magic word "non-zero".

ii. The *algebraic multiplicity* of λ is the number of times $(x-\lambda)$ goes into the characteristic polynomial of T. More explicitly, if the char poly of T is $p(x)$ then $p(x) = (x-\lambda)^a q(x)$ with $q(\lambda) \neq 0$, and the algebraic multiplicity is a. One mark.

The *geometric multiplicity* of λ is the dimension of the space $E_\lambda := \{v \in V : Tv = \lambda v\}$. One mark.

iii. If we write $p(x) = (x-\lambda_1)^{a_1} \cdots (x-\lambda_r)^{a_r}$, which we can do because our base field is the complexes, then $a(\lambda_i) = a_i$ and so $\sum_i a(\lambda_i) = \deg(p(x)) = n$. One mark.

iv. λ is an eigenvalue, so there exists some eigenvector v with $Tv = \lambda v$. Hence $0 \neq v \in E_\lambda$ and hence the dimension of E_λ is strictly positive, so $g(\lambda) > 0$. One mark.

v. The eigenvalues of an upper triangular matrix are just the diagonal entries, so they are 1 and 2. The characteristic polynomial of A is $(x-1)^3(x-2)$ and hence $a(1) = 3$ and $a(2) = 1$. We know that $1 \leq g(2) \leq a(2) = 1$ from lectures, and hence $g(2) = 1$. To compute $g(1)$ we need to do a calculation. Say $v = (a\ b\ c\ d)^t$ is in E_1. Then $Tv = v$ and multiplying this out, we get the equations

$$\begin{aligned} a+b+c+d &= a \\ b+c+d &= b \\ c+d &= c \\ 2d &= d \end{aligned}$$

The last two equations are equivalent to $d = 0$, the second then implies $c = 0$ and the first implies $b = 0$. Conversely any matrix $(a\ 0\ 0\ 0)^t$ is clearly an eigenvector with eigenvalue 1. We deduce that E_1 is 1-dimensional, so $g(1) = 1$.

One mark for getting both a's right, one for $g(2) = 1$, and one for $g(1) = 1$.

vi. An explicit computation shows that the characteristic polynomial of B is $\det(xI - B) = x^3 - 2b^2x$, which factors as $x(x+b\sqrt{2})(x-b\sqrt{2})$. If $b \neq 0$ then the char poly has distinct roots, and hence B is diagonalisable by a result in lectures. If $b = 0$ then the char poly does not have distinct roots but the matrix is diagonal anyway. Hence B is diagonalizable for all $b \in \mathbf{C}$. Two marks.