Imperial College London
M2PM2 Algebra II, Progress Test 3, 7/12/2012.

Name (IN CAPITALS!):..**CID**: ....

Marker's initials:..................

Throughout this test, you may assume any standard results from the course, unless you are explicitly asked to prove them.

Q1. [10 marks]

i. Define the *determinant* $\det(A)$ of an $n \times n$ matrix A.

ii. Prove that if B is the matrix obtained from A by swapping two columns, then $\det(B) = -\det(A)$.

iii. Let σ be a permutation of the set $\{1, 2, 3, \ldots, n\}$ and define an $n \times n$ matrix C as follows: the (i, j)th entry c_{ij} of C is zero unless $j = \sigma(i)$, in which case it is 1. Prove that $\det(C) = \mathrm{sgn}(\sigma)$.

iv. Compute the determinant of

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

v. Let D_n be the $n \times n$ matrix with (i, j)th entry d_{ij} given by the formula: $d_{ij} = 1$ if $|i - j| \leq 1$ and $d_{ij} = 0$ otherwise, so D_n looks like this:

$$\begin{pmatrix} 1 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & \ldots & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & \ldots & 0 & 0 & 0 \\ & & & & \ldots & & & \\ 0 & 0 & 0 & 0 & \ldots & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 \end{pmatrix}$$

If $\delta_n = \det(D_n)$ then prove that $\delta_{n+2} = \delta_{n+1} - \delta_n$ for $n \geq 1$.

vi. With notation as in the previous part, is the matrix D_5 invertible? What about the matrix D_6?

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Q2. [10 marks]

Reminder: you may assume any standard results from the course, unless you are asked to prove them.

Let V be a finite-dimensional vector space over the complex numbers, and let $T : V \to V$ be a linear map.

i. Define what it means for $v \in V$ to be an *eigenvector* for T, with eigenvalue λ.

ii. If λ is an eigenvalue of T then define the *algebraic multiplicity* $a(\lambda)$ and the *geometric multiplicity* $g(\lambda)$ of λ.

iii. Let the eigenvalues of T be $\lambda_1, \lambda_2, \ldots, \lambda_r$, with the λ_i all distinct. Prove that the sum of the algebraic multiplicities $a(\lambda_i)$ is equal to n.

iv. Prove that if λ is an eigenvalue of T then $g(\lambda) > 0$.

v. Let A denote the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

and let $T : \mathbf{C}^4 \to \mathbf{C}^4$ denote the corresponding linear map. What are the eigenvalues of T? For each eigenvalue, compute its algebraic and geometric multiplicity.

vi. Let B denote the matrix

$$\begin{pmatrix} 0 & b & 0 \\ b & 0 & b \\ 0 & b & 0 \end{pmatrix}$$

with b a complex number. For which values of $b \in \mathbf{C}$ is the matrix B diagonalizable over the complexes?