Imperial College London
M2PM2 Algebra II, Progress Test 2, 16/11/2012, solutions.

Q1.

i. G is abelian if $xy = yx$ for all $x, y \in G$. One mark.

ii. ϕ is a group homomorphism if $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in G$. One mark.

iii. For $x, y \in G$ we have $\phi(xy) = \phi(x)\phi(y)$ iff $(xy)^{-1} = x^{-1}y^{-1}$ iff $y^{-1}x^{-1} = x^{-1}y^{-1}$. Taking inverses of both sides we see that this is true iff $xy = yx$. Hence $\phi(x)\phi(y) = \phi(xy)$ iff $xy = yx$. Hence $\phi(x)\phi(y) = \phi(xy)$ for all x, y iff $xy = yx$ for all x, y, and so ϕ is a homomorphism if and only if G is abelian. Two marks.

iv. For $x, y \in G$ we have $\psi(xy) = \psi(x)\psi(y)$ iff $(xy)^2 = x^2y^2$ iff $xyxy = x^2y^2$, and cancelling x on the left and y on the right we deduce that this is true iff $yx = xy$. So as in the previous part, ψ is a group hom iff G is abelian. Two marks.

v. If $G = S_3 \cong D_6$ – the only non-abelian group of order 6, then $\alpha(g) = e_G$ for all $g \in G$, because the order of the element divides the order of the group! Hence α is a group homomorphism – it's the trivial group homomorphism. Two marks.

Imperial College London
M2PM2 Algebra II, Progress Test 2, 16/11/2012, solutions.

Q2.

i. N is a normal subgroup if it's a subgroup satisfying $x^{-1}Nx = N$ for all $x \in G$. One mark (and of course it's also fine to say $xNx^{-1} = N$ for all $x \in G$).

ii. We know $mNm^{-1} = N$, and multiplying both sides on the right by m we deduce that $mN = Nm$. So nm as in the question is, by definition, in Nm and hence it's in mN, so it can be written as mn' for some $n' \in N$. One mark.

iii. To check H is a subgroup we need to check the identity is in, and that H is closed under products and inverses.

Identity: $e = e_G \in M$ and $e \in N$ so $e = e.e \in MN = H$. One mark.

Products: $if m_1n_1$ and $m_2n_2 \in H$ (with $m_i \in M$, $n_i \in N$) then the product is $m_1n_1m_2n_2$, and applying the previous part to n_1m_2 we deduce that $n_1m_2 = m_2n_3$ for some $n_3 \in N$. Hence $m_1n_1m_2n_2 = m_1m_2n_3n_2 = (m_1m_2)(n_3n_2) \in H$. One mark.

Inverse: if $mn \in H$ then $(mn)^{-1} = n^{-1}m^{-1}$ and $n^{-1} \in N$ so again we can apply the previous part to write this as $m^{-1}n'$ which is in H as $m^{-1} \in M$. One mark.

[Anyone who does fancier checks for subgroups should also get full marks if the answer is right – for example you can check that H is a subgroup by checking it's non-empty and for all $x, y \in H$ we have $xy^{-1} \in H$.]

Imperial College London
M2PM2 Algebra II, Progress Test 2, 16/11/2012, solutions.

Q3.

i. x and y are *conjugate* if there exists $g \in G$ such that $gxg^{-1} = y$. One mark.

ii. Clearly the identity is in H. If $g, h \in H$ then $x(gh) = (xg)h = (gx)h = g(xh) = g(hx) = (gh)x$ so $gh \in H$. Finally if $g \in H$ then $xg = gx$ and hence (multiply on the left by x^{-1} and on the right by x) we see $gx^{-1} = x^{-1}g$, so $x^{-1} \in H$. Two marks.

iii. $Hg_1 = Hg_2$ iff $g_1g_2^{-1} \in H$. Note also that $g_1^{-1}xg_1 = g_2^{-1}xg_2$ iff (multiply on the left by g_1 and on the right by g_2^{-1}) $xg_1g_2^{-1} = g_1g_2^{-1}x$ iff $g_1g_2^{-1} \in H$. Two marks.

iv. The conjugacy class of x is the set $\{g^{-1}xg : g \in G\}$. Let's define a map from the set of right cosets Hg for H in G, to the conjugacy class, by mapping Hg to $g^{-1}xg$. This is well-defined by the previous part, and injective by the previous part. It's also clearly surjective – as Hg maps to $g^{-1}xg$. Hence it's a bijection and the size of the conjugacy class is equal to the number of right cosets (and hence divides n as the number of these cosets is $|G|/|H|$). Two marks.