Imperial College London
M2PM2 Algebra II, Progress Test 2, 15/11/2012.

Name (IN CAPITALS!):..**CID**: ....

Marker's initials:..................

Q1. [**8 marks**]

i. Let G be a group. Define what it means for G to be *abelian.*

ii. If G and H are groups, and $\phi : G \to H$ is a map between them, then define what it means for ϕ to be a *group homomorphism.*

For the rest of this question we will consider the case $G = H$, and analyse certain maps $\phi : G \to G$.

iii. Define $\phi : G \to G$ by $\phi(g) = g^{-1}$. Prove that ϕ is a group homomorphism if and only if G is abelian.

iv. Define $\psi : G \to G$ by $\psi(g) = g^2$. Prove that ψ is a group homomorphism if and only if G is abelian.

v. Give an example of a group G that is not abelian, but which has the following property: if $\alpha : G \to G$ is defined by $\alpha(g) = g^6$, then α is a group homomorphism.

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Q2. [5 marks]

Let G be a group.

i. Define what it means for a subgroup $N \subseteq G$ to be a *normal subgroup*.

ii. Now say N is a normal subgroup of G. Prove that if $m \in G$ and $n \in N$, then there exists an element $n' \in N$ such that $nm = mn'$.

iii. Now say M and N are subgroups of G, and that N is normal. Let H denote the set of products $\{mn : m \in M,\ n \in N\}$. Prove that H is also a subgroup of G.

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Q3. [7 marks]

Let G be a group.

i. Define what it means for elements x and $y \in G$ to be *conjugate*.

Now say G is a group and fix $x \in G$. Consider the following subset H of G defined thus:

$$H = \{g \in G \ : \ xg = gx\}.$$

In words, H is the set of elements of G that commute with x.

ii. Prove that H is a subgroup of G.

iii. Prove that if g_1 and g_2 are elements of G, then $Hg_1 = Hg_2$ if and only if $g_1^{-1}xg_1 = g_2^{-1}xg_2$.

iv. Deduce that if G is a finite group of size n, then the size of the conjugacy class of x is equal to the number of right cosets of H in G – and in particular it divides n.