Imperial College London
M2PM2 Algebra II, Progress Test 1, 26/10/2012, solutions.

Q1. Mark scheme: +0.5 for every correct answer, −0.5 for every incorrect one, and 0 for no answer. Working not needed, but I've explained the arguments in the answers.

i. F (e.g. cyclic group of order 6 has no subgroup of order 4 by Lagrange – note that logically speaking one has to verify that there *is* a group of order 6 in order to answer this question!)
ii. F (D_8 and C_8: one is abelian, the other isn't).
iii. F (consider the infinite group of all isometries of the plane; it contains a rotation by an angle of $\pi/2$).
iv. F (f may not be a bijection: for example let G and H be any groups and define $f(g) = e$ for all $g \in G$).
v. T (signature of a c-cycle is $(-1)^{c-1}$ from lectures).
vi. T (isomorphism is an equivalence relation, from example sheet).
vii. F (for silly reasons: S_1 has only one element).
viii. T ($\mathrm{sgn}(g^3) = \mathrm{sgn}(g)^3$ and check both cases)
ix. T (for example $S_{2,000,000}$ contains well over a million transpositions).
x. T (for example X and Y can just be the subgroups consisting only of the identity.)

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Q2.

i. By Lagrange, the order of H divides 10 so it's 1, 2, or 5 (or 10 but that's ruled out by the question). Any group of order 1 or of prime order is cyclic (again by Lagrange) so we're done. Two marks.

ii. D_{10} is a subgroup, and all other subgroups are cyclic so are generated by one element. We hence go through all the elements. We have the identity subgroup of course. From lectures we have a subgroup $\langle\rho\rangle$ of order 5, and it is not hard to check that $\langle\rho\rangle = \langle\rho^2\rangle = \langle\rho^3\rangle = \langle\rho^4\rangle$ (because any non-trivial element in a cyclic group of prime order p must have order p so must generate everything). The other five elements of D_{10} are all reflections, so we get five subgroups of order 2.

Conclusion: $\{e\}$, D_{10}, $\langle\rho\rangle$ and five subgroups of order 2, giving eight subgroups. Four marks for this list.

All the groups of order 2 are cyclic and hence isomorphic, so up to isomorphism there are only four subgroups – there can't be any more isomorphisms because C_1, C_2, C_5 and D_{10} all have different orders. Two marks.

iii. We know $\sigma\rho\sigma = \rho^{-1}$, and raising this to the power n gives $\sigma\rho^n\sigma = \rho^{-n}$. Hence the element is $\rho(\sigma\rho^2\sigma)(\rho^3)(\sigma\rho^4\sigma) = \rho\rho^{-2}\rho^3\rho^{-4} = \rho^{-2} = \rho^3$. Two marks.

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Q3.

i. This is true: $\operatorname{sgn}(gh) = \operatorname{sgn}(g)\operatorname{sgn}(h) = \operatorname{sgn}(h)\operatorname{sgn}(g) = \operatorname{sgn}(hg)$. One mark.

ii. The possibilities are: the identity, (2) (a two-cycle), (3) (3-cycle), (4) (4-cycle) and (2)(2) (two disjoint two-cycles). A mark. Knowing the cycle shape is enough to determine the signature, by the formula $\operatorname{sgn}(g) = (-1)^{\sum_i (c_i - 1)}$ for g a disjoint product of cycles of lengths c_i. Applying this to the possibilities above, we see that the identity, (3) and (2)(2) are even, and (2) and (4) are odd. Two marks.

iii. We do a case-by-case check. The identity is (1 2)(1 2). A three-cycle is a product of two 2-cycles by a result in lectures – or note $(a\ b\ c) = (a\ c)(a\ b)$. Finally (2)(2) is clearly a disjoint product of two 2-cycles. Three marks.

iv. It's not true – a product of two 2-cycles can clearly only move at most 4 elements, but a 5-cycle has signature $(-1)^4 = 1$ and moves five elements. Three marks.