M2PM2 test 2, 19/11/13, solutions and mark scheme.

Q1.

i. The group $C_2 \times C_4 \times C_{12}$ has $2 \times 4 \times 12 = 8 \times 12 = 96$ elements. One peasy mark. If (a, b, c) is a general element, then it has order 6 iff $(a, b, c)^6 = (1, 1, 1)$ iff $a^6 = b^6 = c^6 = 1$. Let's figure out the solutions to these equations. We know $a \in C_2$ so $a = \pm 1$ and $a^6 = 1$ is always true, so there are two possibilities for a. Now $b^4 = 1$ so $b \in \{\pm 1, \pm i\}$ and only ± 1 are solutions to $b^6 = 1$, so there are two possibilities for b. But c is a general 12th root of unity, and every 6th root of unity is a 12th root of unity, so all six 6th roots of unity work for c, giving 6 choices. Hence in total we have $2 \times 2 \times 6 = 24$ elements of order dividing 6. Two marks.

ii. It is. Remember that the group law is addition! We need to check that for all $g \in \mathbf{R}$, $-g + \mathbf{Z} + g = \mathbf{Z}$. But $\mathbf{R}$ is abelian so $-g + \mathbf{Z} + g = \mathbf{Z} - g + g = \mathbf{Z}$. One mark. (even just saying "$\mathbf{R}$ is abelian so it follows from a result in the course" is fine).

iii. Clearly $M \times N$ is a subset of $G \times H$: it's the subset consisting of (g, h) such that $g \in M$ and $h \in N$. To check it's a subgroup, we need to check (e_G, e_H) is in (which it is, because subgroups M and N contain their respective identities), that products of things in $M \times N$ are in $M \times N$ (which is true because if (m_1, n_1) and $(m_2, n_2) \in M \times N$ then so is $(m_1, n_1)(m_2, n_2) = (m_1 m_2, n_1 n_2)$), and that inverses of things in $M \times N$ are in – but $(m, n)^{-1} = (m^{-1}, n^{-1}) \in M \times N$, so we're home. One mark, and give it the moment they indicate that they know how to check a subset of a group is a subgroup and say something sensible about how the check works in this situation.

Now say M and N are both normal (in G and H respectively). Then indeed it's true that $M \times N$ is normal in $G \times H$. For if $(g, h) \in G \times H$ then a typical element of $(g, h)^{-1}(M \times N)(g, h)$ is $(g, h)^{-1}(m, n)(g, h)$ (with $m \in M$ and $n \in N$), which is easily checked to be $(g^{-1}mg, h^{-1}nh)$. Normality of M implies $g^{-1}mg \in M$, and similarly $h^{-1}nh \in N$. So $(g, h)^{-1}(M \times N)(g, h) \subseteq M \times N$, and by a result from the course this is enough (because it's true for an arbitrary (g, h)). Two marks.

iv. This is not true. For example if $G = H = C_3$ and $x = e^{2\pi i/3}$ is a generator of G, then $H = \langle x^{-1} \rangle$ and hence there's an isomorphism $G \to H$ sending x to x^{-1}. Other examples abound. For example, I noted on the example sheet that if $x \in G = H$ then the map $G \to H$ sending $g \in G$ to $x^{-1}gx$ is an isomorphism, and this map is often not the identity map if G is non-abelian. Two marks.

v. σ is a product of a bunch of transpositions: it's $(1\ 2013)(2\ 2012)(3\ 2011)\ldots$. How does this sequence finish? The last two elements to be swapped are 1006 and 1008, and then 1007 gets sent to itself. So that makes 1006 transpositions, and the permutation will hence have signature $(-1)^{1006} = +1$. One mark.

Q2.

i. $\det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\,\sigma(i)}$ or equivalent. One mark.

ii. We know $a_{ij} = 0$ if $i > j$. Say $\prod_i a_{i\,\sigma(i)} \neq 0$. Then all the $a_{i\,\sigma(i)}$ had better be non-zero. In particular $\sigma(n)$ had better be n, because if it's less than n then $a_{n\,\sigma(n)} = 0$. Then $\sigma(n-1)$ had better be $n-1$ because it can't be n and it has to be at least $n-1$. And so on. So σ must be the identity and the result follows easily. Two marks. One mark for "expand along the bottom row/first column and proceed by induction", because that's not a proof directly from the definition, that's a proof assuming a tricky theorem.

iii. If $B = (b_{ij}) = A^T$ then $b_{ij} = a_{ji}$, so

$$\begin{aligned}\det(B) &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_j b_{j\,\sigma(j)} \\ &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_j a_{\sigma(j)\,j} \\ &= \sum_\sigma \operatorname{sgn}(\sigma) \prod_i a_{i,\sigma^{-1}(i)} \\ &= \sum_\sigma \operatorname{sgn}(\sigma^{-1}) \prod_i a_{i\,\sigma^{-1}(i)} \\ &= \det(A).\end{aligned}$$

Two marks.

iii. It's complete nonsense. For example if $n = 2$ and $A = B = I_n$ are the identity matrix then by part (ii) we see that $\det(A) = \det(B) = 1$ and $\det(A + B) = 2^n \neq 2$. Generous two marks.

iv. The first and last rows of this matrix are the same, so the determinant is zero by a lemma from the course. One mark.

v. Let $A_n = (a_{ij})$ denote the $n \times n$ matrix with $a_{ij} = 1$ if $i = j + 1$ or $i = j - 1$, and $a_{ij} = 0$ otherwise. Expanding down the first column, and then along the first row, we pick up a minus sign and deduce $\det(A_n) = -\det(A_{n-2})$. The question asks us what the determinant of A_{100} is, and an explicit calculation shows that A_2 has determinant -1, so $\det(A_4) = +1$, $\det(A_6) = -1$ and so on; finally, 100 is a multiple of 4, so $\det(A_{100}) = +1$. Two marks.