M2PM2 Algebra II, Progress Test 2, 25/11/2014.

This test has two questions, Q1 and Q2. They are worth the same number of marks.

Q1. In this question you may assume any results from the course that you need, including the fundamental fact that a finite abelian group is isomorphic to a product of cyclic groups. You can't assume results from problem sheets though, unless you supply proofs.

i) Up to isomorphism, how many abelian groups of order 12 are there? Justify your answer.

ii) How many different isomorphisms are there from C_6 to C_6? Justify your answer.

iii) Construct a group of order greater than 100 such that every element has order 1 or 2. Justify your answer.

iv) Let G be a group, let $g \in G$ be a fixed element of this group, and define a map $\phi : G \to G$ by $\phi(x) = g^{-1}xg$. Prove that ϕ is an isomorphism.

Q2. In this question you may also assume any results from the course that you need.

i) Give an example of groups G and H, such that there is an element $g \in G$ of order 4, an element $h \in H$ of order 2, and a group homomorphism $\phi : G \to H$ sending g to h. Justify your answer.

ii) Do there exist groups G and H, and elements $g \in G$ of order 4 and $h \in H$ of order 3, and a group homomorphism $\phi : G \to H$ sending g to h? Justify your answer.

iii) Let G be a group, and let N and M be normal subgroups of G. Prove that $M \cap N$ is a normal subgroup of G (you may assume that $M \cap N$ is a subgroup of G).

iv) Let G be a group, and let M and N be normal subgroups of G. Let X be the set $\{mn : m \in M, n \in N\}$. Prove that X is a subgroup of G. Hint: prove the following lemma first: if $m \in M$ and $n \in N$ then $nm \in X$.