M2PM2 Algebra II, Progress test 1, 31/10/2014, solutions.

Q1.

i. False – can easily check that $\rho^{-1} = \rho^4$, and that ρ^2 is another element of order 5. One mark.

ii. $\rho\sigma\rho^2 = \rho(\sigma\rho)\rho = \rho(\rho^{-1}\sigma)\rho = \sigma\rho = \rho^{-1}\sigma = \rho^4\sigma$. Both $m = 4$ and $m = -1$ are fine. Two marks.

iii. All the ρ^i, $1 \leq i \leq 4$, generate the same subgroup, namely the subgroup of rotations $\langle\rho\rangle$. The five reflections generate five distinct subgroups of order 2, so we have these five, plus the subgroup of rotations, plus the trivial subgroup, giving a total of 7. Three marks.

iv. Let's say the reflections are $g = \rho^i\sigma$ and $h = \rho^j\sigma$ for some $i \neq j$. Then H contains g and h, so it also contains $r = gh^{-1} = \rho^i\sigma(\sigma^{-1}\rho^{-j}) = \rho^{i-j}$. Now becasue $g \neq h$ we know $\rho^i \neq \rho^j$, and hence $r \neq e$ is a non-trivial rotation, so H contains $\langle r\rangle = \langle\rho\rangle$; this gives five elements of H, and g, h give two more which is 7, and now we're home because the size of H must divide 10 and so it's equal to 10. Four marks.

Q2.

i. Say $\alpha : G \to H$ and $\beta : H \to K$ are isomorphisms. Set $\gamma = \beta \circ \alpha : G \to K$. The claim is that γ is an isomorphism $G \to K$. It's certainly a bijection (by 1st year) so the issue is whether it satisfies $\gamma(xy) = \gamma(x)\gamma(y)$. But it does, because

$$\begin{aligned}\gamma(xy) &= \beta(\alpha(xy)) \text{ (by definition)}\\ &= \beta(\alpha(x)\alpha(y)) \text{ (α is an isom)}\\ &= \beta(\alpha(x))\beta(\alpha(y)) \text{ (β is an isom)}\\ &= \gamma(x)\gamma(y) \text{ (by definition)}\end{aligned}$$

Two marks for this.

ii. D_{12} contains an element of order 6 (a rotation), but A_4 is (by lectures) the elements of S_4 which are either the identity, a 3-cycle, or the product of two 2-cycles, so every element has order 1, 2 or 3. So they're not isomorphic. Two marks.

iii. It is a bijection (because $f \circ f$ is the identity) but it's not an isomorphism. Let's say it was. Then we would be able to deduce that $f(xy) = f(x)f(y)$ for all $x, y \in S_3$, and hence $(xy)^{-1} = x^{-1}y^{-1}$. Taking inverses we deduce $xy = yx$ (because yx is clearly the inverse of $x^{-1}y^{-1}$) for all x and y, but this is not true in S_3 because, for example, $x = (1\,2)$ and $y = (2\,3)$ are easily checked to not commute. Four marks for that toughie.

iv. The cycles have signature $+1, -1, +1$ by results in lectures, so the element has signature -1. One mark.

v. How about $(1\,2)(3\,4)$. One mark.