M2PM2 Algebra II, Problem Sheet 8

Like the last sheet, this sheet runs over the page.

1. Define a relation on $n \times n$ matrices by $A \sim B$ if and only if there exists an invertible $n \times n$ matrix P such that $B = P^{-1}AP$. Prove that $\sim$ is an equivalence relation. Remark: we say "A is similar to B" if $A \sim B$.

2. Let E, F be the following bases of $\mathbb{R}^2$: $E = \{(1,0),\ (0,1)\}$, $F = \{(1,3),\ (2,5)\}$.

(i) Find the change of basis matrix P from E to F.

(ii) Find the change of basis matrix Q from F to E.

(iii) Check that $Q = P^{-1}$.

(iv) Let $v = (a, b) \in \mathbb{R}^2$. Write down the column vectors $[v]_E$ and $[v]_F$, and check that $[v]_E = P[v]_F$.

(v) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (2x_2, 3x_1 - x_2)$. Check that $[T]_F = P^{-1}[T]_E P$.

3. Find the determinants of the following linear transformations T:

(i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ -x_1 - 3x_3,\ x_2 + x_3).$$

(ii) Let V be the vector space of polynomials of degree at most 3 over $\mathbb{R}$, and define $T : V \to V$ by $T(p(x)) = p(1 + x) - p'(1 - x)$ for all $p(x) \in V$.

(iii) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$, let $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, and define $T : V \to V$ by $T(A) = MA$ for all $A \in V$. Hint: It's not often we think of V as a vector space, but it is (it's isomorphic to $\mathbb{R}^4$) and you might want to consider a basis of V consisting of $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $e_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ – note the order of the e_i; this makes it work a bit better – trust me.

4. For T as in each of parts (ii) and (iii) of Q3, find the eigenvalues of T, and find a basis for each eigenspace. Is there a basis B of V such that the matrix $[T]_B$ is diagonal ?

5. (a) For each of the following linear transformations $T : V \to V$, find its eigenvalues, find their algebraic and geometric multiplicities, and determine whether V has a basis consisting of eigenvectors of T:

(i) $V = \mathbb{R}^3$, $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3,\ -4x_2 + 6x_3,\ -3x_2 + 5x_3)$.

(ii) V the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x + 1) - p(x) - p(x - 1))$ for all $p(x) \in V$.

(b) For which values of a, b, c is the matrix

$$A = \begin{pmatrix} -1 & a & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalisable (i.e. $\exists\ P$ such that $P^{-1}AP$ is diagonal) ?

6. Let V be an n-dimensional vector space over $\mathbb{R}$, $n \geq 1$. Define $GL(V)$ to be the set of all invertible linear transformations from $V \to V$.

(i) Prove that $GL(V)$ is a group under composition.

(ii) Prove that the function $\det : GL(V) \to (\mathbb{R}^*, \times)$ is a surjective homomorphism.

(iii) Prove that $GL(V) \cong GL(n, \mathbb{R})$. (Recall that $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices over $\mathbb{R}$, under matrix multiplication.)

7.‡What is the determinant of

$$\begin{pmatrix} \cos(1) & \cos(2) & \cos(3) & \cos(4) \\ \cos(5) & \cos(6) & \cos(7) & \cos(8) \\ \cos(9) & \cos(10) & \cos(11) & \cos(12) \\ \cos(13) & \cos(14) & \cos(15) & \cos(16) \end{pmatrix}?$$

Why? (cosines are in radians)