KMB, 25th November 2014

M2PM2 Algebra II Problem Sheet 7

Matrices take up a lot of room on the page, so this example sheet takes up two sides.

1. Calculate the determinants of the following matrices.

a) $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$ b) $\begin{pmatrix} -2 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ -3 & 1 & 7 & 0 & 0 \\ 4 & 0 & 2 & -1 & 0 \\ 5 & 6 & 7 & 5 & 3 \end{pmatrix}$

c) $\begin{pmatrix} m & 0 & 0 & a & b \\ n & 0 & e & d & c \\ p & 0 & 0 & 0 & k \\ r & \ell & f & g & j \\ h & 0 & 0 & 0 & t \end{pmatrix}$ d) $\begin{pmatrix} 1 & 2 & -2 & 3 & 5 \\ -4 & 2 & 4 & 2 & 1 \\ 0 & 0 & 0 & 5 & 5 \\ 0 & 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 & 9 \end{pmatrix}$

2. For a real number α define

$$A(\alpha) = \begin{pmatrix} 1 & \alpha & 0 & -1 \\ 1 & 1 & 0 & -1 \\ 2 & \alpha & 1 & -1 \\ -1 & \alpha & 1 & 1 \end{pmatrix}$$

(a) Find the determinant of $A(\alpha)$.

(b) Find a value α_0 of α such that the system $A(\alpha_0)x = 0$ has a nonzero solution for $x \in \mathbb{R}^4$.

(c) Prove that when $\alpha < \alpha_0$, there is no real 4×4 matrix B such that $B^2 = A(\alpha)$.

3. Let A_n be the $n \times n$ matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix}$$

(a) Prove that $|A_n| = n + 1$.

4. Let B_n be the $n \times n$ matrix

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ -1 & 1 & 1 & \dots & 1 & 1 & 1 \\ 0 & -1 & 1 & \dots & 1 & 1 & 1 \\ & & & \dots & & & \\ 0 & 0 & 0 & \dots & -1 & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & -1 & 1 \end{pmatrix}$$

Prove that $|B_n| = 2^{n-1}$.

5. Let $A = \begin{pmatrix} B & C \\ \mathbf{0} & D \end{pmatrix}$, where B is $s \times s$, D is $t \times t$, C is $s \times t$, and $\mathbf{0}$ is the $t \times s$ zero matrix. Prove that $\det(A) = \det(B)\det(D)$.

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6. Let's finish the proof that $|AB| = |A|.|B|$. Let A, B be $n \times n$ matrices.

(a) Prove that if $|A| = 0$ then $|AB| = 0$.

(b) Prove that if $|B| = 0$ then $|AB| = 0$.

Note: you may NOT assume the result $|AB| = |A|\,|B|$ from lectures, because this question is part of the proof *of that result! But you may assume the result in lectures that says a matrix is invertible iff it has nonzero determinant.*

7. Let's also prove the following result about elementary matrices, which I used a couple of times. With notation for elementary matrices as in lectures:

(a) Prove $|A_i(r)| = r$, $|B_{ij}| = -1$ and $|C_{ij}(r)| = 1$.

(b) Prove $A_i(r)^{-1} = A_i(r^{-1})$, $B_{ij}^{-1} = B_{ij}$ and $C_{ij}(r)^{-1} = C_{ij}(-r)$.

8. Express $\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ 3 & 8 & 7 \end{pmatrix}$ as a product of elementary matrices.

9. For $n \times n$ matrices A, B, write $A \sim B$ to mean that B can be obtained from A by a sequence of elementary row operations.

Prove that $A \sim B$ if and only if $B = E_1 \dots E_k A$, where each E_i is an elementary matrix. Deduce that the relation $\sim$ is an equivalence relation.

10.‡ Say A and B are $n \times n$ matrices with real entries, and $A^2 + B^2 = AB$. If $AB - BA$ is invertible, prove that n is a multiple of 3.