M2PM2 Algebra II Problem Sheet 6

- **1.** (a) Let G be a cyclic group, and let N be a subgroup of G. Prove that $N \triangleleft G$ and the factor group G/N is also cyclic.
- (b) Let $n \in \mathbb{N}$, and let $n\mathbb{Z} = \langle n \rangle$ be the cyclic subgroup of $(\mathbb{Z}, +)$ generated by n. Use the First Isomorphism Theorem to prove that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $(\mathbb{Z}_n, +)$, the group of residue classes modulo n under addition.
- (c) Say G and H are groups, and $\phi: G \times H \to H$ is defined by $\phi(g,h) = h$. Prove that ϕ is a homomorphism, and prove that the kernel of ϕ is isomorphic to G.
- (d) Find a non-abelian group G, and a normal subgroup N of G, such that $G/N \cong C_2 \times C_2 \times C_2$.
- **2.** (i) Prove that if there is a surjective homomorphism from S_n onto C_r , then r must be 1 or 2.
- (ii) (quite tricky!) Prove that if N is a normal subgroup of S_n such that the factor group S_n/N is cyclic, then $N = S_n$ or A_n .
- **3.** For each of the following groups G, find all groups H (up to isomorphism) such that there is a surjective homomorphism from G to H:

$$G = D_8$$
, $G = D_{12}$, $G = S_4$.

Slight warning: S_4 is tricky – but satisfying!

4. In lectures we defined dihedral groups by looking at symmetry groups of certain 2-dimensional shapes. These tricks are not confined to two dimensions however. In this question we investigate the symmetries of a regular tetrahedron. The general story in 3 dimensions is the same as the 2-dimensional case: an isometry is a map $f: \mathbb{R}^3 \to \mathbb{R}^3$ such that d(x,y) = d(f(x), f(y)), where d is the usual distance function on \mathbb{R}^3 : d(x,y) is the distance from x to y.

Examples of isometries $\mathbb{R}^3 \to \mathbb{R}^3$: rotation about a line, reflection in a plane.

Now let Π denote a regular tetrahedron, and let's label the vertices 1, 2, 3 and 4.

- (a) Prove that there are at least 12 rotations mapping Π to Π . Hint: check that as well as the identity, there are 8 rotations of order 3 that fix a vertex, and 3 rotations of order 2 that fix the midpoints of two opposite edges.
- (b) Prove that there are exactly 12 rotations mapping Π to Π . Hint: there are four faces, and each face can be orientated in three ways. Look at how we did this sort of thing in the 2-d case.
- (c) Prove that the group of rotations that preserve Π is isomorphic to A_4 . Hint: consider permutations of the vertices.
- (d) Prove that the full symmetry group of the tetrahedron is isomorphic to S_4 . Hint: check that there is a reflection that switches two vertices and fixes the other two.
- **5.** Let R be the set of rotations in the symmetry group of a cube.
- (a) By writing down as many rotation symmetries as you can think of, show that $|R| \ge 24$.
- (b) By counting the possibilities for the 'bottom' face and the 'front' face, show that $|R| \leq 24$.
- (c) By considering each rotation as a permutation of the 4 long diagonals of the cube, show that $R \cong S_4$.
- **6.**[‡] What do you think the group of rotations of an octahedron is? A dodecahedron? An icosahedron? A 10-dimensional hypercube? A hecatonicosachoron?