KMB, 10th Nov 2014.

M2PM2 Algebra II, Problem Sheet 5

1. Which of the following functions ϕ are group homomorphisms? For those which are group homomorphisms, find Im ϕ and Ker ϕ.

$$\phi : C_{12} \to C_{12} \text{ defined by } \phi(x) = x^3 \ \ \forall x \in C_{12}$$
$$\phi : S_4 \to S_4 \text{ defined by } \phi(x) = x^3 \ \ \forall x \in S_4$$
$$\phi : (\mathbb{Z}, +) \to (\mathbb{Z}_n, +) \text{ defined by } \phi(x) = [x]_n \ \ \forall x \in \mathbb{Z}$$
$$\phi : (\mathbb{R}_{>0}, \times) \to (\mathbb{R}_{>0}, \times) \text{ defined by } \phi(x) = \sqrt{x} \ \ \forall x \in \mathbb{R}_{>0}$$
$$\phi : (\mathbb{Z}_6, +) \to (\mathbb{Z}_7, +) \text{ defined by } \phi([x]_6) = [x]_7 \ \ \forall [x]_6 \in \mathbb{Z}_6$$
$$\phi : (\mathbb{Z}_6, +) \to (\mathbb{Z}_7^*, \times) \text{ defined by } \phi([x]_6) = [2^x]_7 \ \ \forall [x]_6 \in \mathbb{Z}_6$$

Recall the notation: $[x]_n$ stands for the residue class of x modulo n, and $\mathbb{R}_{>0}$ stands for the set of positive real numbers.

2. Let G be a group.

(a) We say that elements $x, y \in G$ are *conjugate* (or more precisely are *conjugate in* G) if there exists $g \in G$ with $g^{-1}xg = y$. Prove that conjugacy (which means "being conjugate") is an equivalence relation.

(b) The equivalence classes in (a) are called *conjugacy classes*. Prove that a subgroup H of G is normal iff it is a union of conjugacy classes.

3. (a) Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5) \in S_5$. Prove that σ is conjugate to σ^2 in S_5.

(b) Now note that $\mathrm{sgn}(\sigma) = +1$ so $\sigma \in A_5$. Is σ conjugate to σ^2 in A_5? [hint: if $x\sigma x^{-1} = \sigma^2$ then there are only five possibilities for $x(1)$, and $x(1)$ determines x because $x\sigma x^{-1} = \sigma^2$].

4. Let G be a group, and suppose M and N are normal subgroups of G. Show that $M \cap N$ is a normal subgroup of G.

5. Let G be a group, let $g \in G$ be an element of this group, and define a homomorphism $\phi : G \to G$ by $\phi(x) = g^{-1}xg$. Prove that ϕ is an isomorphism. Give an example to show that ϕ may not be the identity map.

6. Let $G = D_{2n} = \{e, \rho, \ldots, \rho^{n-1}, \sigma, \rho\sigma, \ldots, \rho^{n-1}\sigma\}$, where $n \geq 3$ and ρ, σ satisfy the usual equations $\rho^n = \sigma^2 = e$, $\sigma\rho = \rho^{-1}\sigma$. It is not hard to prove by induction that $\sigma\rho^k = \rho^{-k}\sigma$ for all integers k (prove it if you're interested).

(a) Let r be a fixed integer. Prove that the cyclic subgroup $\langle \rho^r \rangle$ is a normal subgroup of D_{2n}.

(b) Let r be a fixed integer. Prove that $\langle \rho^r \sigma \rangle$ is not a normal subgroup of D_{2n}.

7. Let p be a prime number greater than 2.

(a) Prove that the dihedral group D_{2p} has exactly three different normal subgroups.

(b) Find all groups H (up to isomorphism) such that there is a surjective homomorphism from D_{2p} onto H.

8. Does there exist a surjective homomorphism

(i) from C_{12} onto C_4 ?

(ii) from C_{12} onto $C_2 \times C_2$?

(iii) from D_8 onto C_4 ?

(iv) from D_8 onto $C_2 \times C_2$?

Give reasons for your answers.

9. (a) Show that if G is an abelian group, and N is a subgroup of G, then $N \lhd G$ and the factor group G/N is abelian.

(b) Give an example of a non-abelian group G with a normal subgroup N such that both N and G/N are abelian.

(c) Give an example of a group G with subgroups M, N such that $N \lhd G$ and $M \lhd N$, but M is not normal in G.

10.‡ Say G and H are finite groups, and there exists a surjective group homomorphism from $G \times G$ to $H \times H$. Must there exist a surjective group homomorphism from G to H?