KMB, 20/10/14

M2PM2 Algebra II Problem Sheet 2

Questions marked with ‡ are challenge problems. They are probably harder than most exam questions and can be safely ignored unless you're looking for a challenge. No answers will be supplied for the ‡ questions.

1. Prove that if $f : G \to H$ is an isomorphism, and $f(g) = h$ then $f(g^n) = h^n$ for all $n \geq 1$.

2. Prove the axioms for an equivalence relation hold for isomorphism of groups, i.e. prove that $G \cong G$, that $G \cong H$ implies $H \cong G$ etc etc).

3. Say G, H are groups and $\phi : G \to H$ is an isomorphism.

(a) Prove that $\phi(g^{-1}) = \phi(g)^{-1}$ for all $g \in G$.

(b) Prove that if g has infinite order then so does $\phi(g)$.

4. Which pairs among the following groups are isomorphic ?

$(\mathbb{Q}, +)$
$(\mathbb{Z}, +)$
$(\mathbb{Q}^*, \times)$
$(\mathbb{Q}_{>0}, \times)$ (the positive rationals under mult.)
the subgroup $\langle \pi \rangle$ of $(\mathbb{R}^*, \times)$
the group $(\mathbb{Q} - \{-1\}, *)$, where $a * b = ab + a + b \ \ \forall a, b \in \mathbb{Q} - \{-1\}$

5. (a) Prove that no two of the groups S_5, C_{120} and D_{120} are isomorphic to each other.

(b) Prove that S_3 is isomorphic to D_6.

(c) Prove that $(\mathbb{R}, +)$ is isomorphic to $(\mathbb{R}_{>0}, \times)$, where $\mathbb{R}_{>0}$ is the set of positive real numbers. Is $(\mathbb{Q}, +)$ isomorphic to $(\mathbb{Q}_{>0}, \times)$?

(d) Prove that D_8 has two subgroups of size 4 which are not isomorphic to each other.

6. Let G be a group with the property that $x^2 = e$ for all $x \in G$.

(a) Prove that G must be abelian.

(b) Prove that if G is finite then either $|G| \leq 2$, or $|G|$ is divisible by 4.

7. Let n be a positive integer.

(a) Prove that there are infinitely many groups of size n.

(b) Prove that, up to isomorphism, there are only finitely many groups of size n.

8. (a) Find the signatures of the following permutations g and h in S_9:

$$g = (1\ 2\ 7\ 8)(3\ 9)(4\ 5\ 6), \quad h = (1\ 3)(8\ 9)(2\ 4\ 8\ 9\ 5).$$

(b) List the cycle-shapes of elements of the alternating group A_7.

(c) Calculate the number of elements of order 2 in A_7.

9. Let $g \in S_n$. Show that if g has odd order, then g must be an even permutation.

10.‡ Prove that "the converse of Proposition 2.1 is false". More precisely, can you find two finite groups G and H which are not isomorphic, but which have the same size, are both non-abelian (or both abelian), and have the same number of elements of order k for all k. Hint: this is probably too hard to do by hand. A good way to do this would be to use a computer algebra package such as GAP, which has a database of groups of order at most 2000, and then write a little computer program to go through the list until you find an example! That was how I did this question, at least...