Course:	M2PM2
Setter:	Setter
Checker:	Checker
Editor:	Editor
External:	N/A
Date:	January 8, 2015

BSc and MSci MOCK EXAMINATIONS SOLUTIONS (MATHEMATICS)

January 2015

M2PM2

Algebra II Solutions

Setter's signature	Checker's signature	Editor's signature
. . . .	. . . .	. . . .

BSc and MSci MOCK EXAMINATIONS (MATHEMATICS)

January 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M2PM2

Algebra II Solutions

Date: Xday, xth January 2015 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

This exam contains **TWO** questions and each question contains **TWO** parts. You must answer **ONE** of the two parts for each question.

Calculators may not be used.

This exam is a mock, keep in mind that the correct format in official examinations will be **FOUR** questions of which all **FOUR** will need to be answered.

Part A – Group Theory

1. (a) A homomorphism ϕ from a group G to a group H is a map $\phi : G \to H$ that additionally satisfies

$$\forall g, h \in G, \quad \phi(gh) = \phi(g)\phi(h).$$

The kernel of ϕ is the subset of all elements of G that are sent to the identity in H by ϕ. $[\mathbf{2\ marks}]$

(b) First, show that the kernel is a subgroup:

- Identity: if $h = \phi(e_G)$ then $h^2 = \phi(e_G)^2 = \phi(e_G^2) = h$ so (multiplying by h^{-1}) $h = e_H$.
- Closure: for elements $g_1, g_2 \in \ker(\phi)$, $\phi(g_1 g_2) = \phi(g_1)\phi(g_2) = e_H e_H = e_H$. Hence $g_1 g_2 \in \ker(\phi)$.
- Inverse: for an element $g \in ker(\phi)$, $e_H = \phi(e_G) = \phi(gg^{-1}) = \phi(g)\phi(g^{-1}) = \phi(g^{-1})$. Hence $g^{-1} \in \ker(\phi)$.

Now prove that kernel is normal: we need to check that for $x \in g\ker(\phi)g^{-1}$, $\phi(x) = e_H$. For some $n \in \ker(\phi)$, $x = gng^{-1}$, hence:

$$\phi(x) = \phi(gng^{-1}) = \phi(g)\phi(n)\phi(g^{-1}) = \phi(g)e_H\phi(g^{-1}) = \phi(gg^{-1}) = e_H.$$

Hence $x \in \ker(\phi)$.$[\mathbf{4\ marks}]$

(c) For (a, b) and $(c, d) \in \mathbb{R}^* \times \mathbb{R}^*$:

$$\phi((a, b)(c, d)) = \phi((ac, bd)) = \frac{ac}{bd} = \frac{a}{b}\frac{c}{d} = \phi(a, b)\phi(c, d).$$

Hence ϕ is a homomorphism. $[\mathbf{2\ marks}]$

(d) The kernel:

$$\ker(\phi) = \{(a, b) \in \mathbb{R}^* \times \mathbb{R}^* \mid a/b = 1\}.$$

Hence a general element of $\ker(\phi)$ is of the form (a, a) for $a \in \mathbb{R}^*$. Now the map $\ker(\phi) \to \mathbb{R}^*$ sending (a, a) to a is clearly a bijection, and it's also easily checked to be a homomorphism, so it's an isomorphism. $[\mathbf{3\ marks}]$

(e) By the first isomorphism theorem

$$\frac{\mathbb{R}^* \times \mathbb{R}^*}{\ker(\phi)} \cong Im(\phi).$$

The image of ϕ is $\mathbb{R}^*$ (consider $\phi(a, 1)$ for $a \in \mathbb{R}^*$) hence we are done. [**4 marks**]

(f) Example, the normal subgroup corresponding to the kernel of ξ where

$$\xi(a, b) = a$$

will do, again by the first isomorphism theorem. Explicitly,

$$\ker(\xi) = \{(1, b) \forall b \in \mathbb{R}^*\}.$$

Any other example is fine provided justification is given. [**5 marks**]

2. (a) An abelian group G is a group where all elements commute with each other, i.e.,

$$\forall g_1, g_2 \in G, g_1 g_2 = g_2 g_1.$$

Now, list all abelian groups of size $4036 = 1009 \times 2^2$: by the structure theorem and the fact that $C_a \times C_b \cong C_{ab}$ if a, b are coprime we see that the only groups are

- $C_{4036} \cong C_{1009} \times C_4$
- $C_{2018} \times C_2 \cong C_{1009} \times C_2 \times C_2$

These two groups are not isomorphic to each other, as one contains an element of order 4036 and the other does not (any element of $C_{2018} \times C_2$ has order dividing 2018) [**3 marks**]

(b) These groups are isomorphic. Indeed, if $\rho \in D_{2018}$ is an order 1009 rotation, then the element $r := (\rho, -1)$ of $D_{2018} \times C_2$ has order equal to the LCM of 1009 and 2, which is 2018, and the element $s := (\sigma, 1)$ has order 2; furthermore $s^{-1}rs = (\sigma^{-1}\rho\sigma, -1) = r^{-1}$, so r and s satisfy the right relation for a dihedral group, and hence the product group is indeed dihedral of order 4036. [**5 marks**]

(c)
- Identity (1): $\langle e \rangle$
- Reflections (2018): $\langle \rho^i \sigma \rangle$ for $i \in \{1, 2, \ldots, 2018\}$
- Rotations (3): $\langle \rho \rangle$, $\langle \rho^2 \rangle$ and $\langle \rho^{1009} \rangle$

(the last part is because the only subgroups of C_n, a cyclic group of order n, are C_d for d dividing n). For a total of 2022. [**4 marks**]

(d) First, G is a subgroup, so the number of elements in G divides the number of elements in D_{4036}. That is: $|G| \in \{1,\ 2,\ 4,\ 1009,\ 2018,\ 4036\}$. Also, $|G| \geq 3$ because it contains 2 reflections and the identity, and is divisible by two because it contains a subgroup of size two (spanned by either reflection). This leaves $|G| = 4$ or 2018 or 4036.
Assume, without loss of generality, that the two distinct reflections are σ and $\rho^i\sigma$ with $i \in 1, 2, \ldots, 2017$. Then, by closure, $\rho^i\sigma\sigma = \rho^i \in G$. By previous question, ρ^i generates a cyclic subgroup of $\langle \rho \rangle$ which must be isomorphic to either C_2, C_{1009} or C_{2018}. We can't have $\langle \rho^i \rangle \cong C_2$ because this would imply that G contained the rotation of order 2, which is not allowed. So $|\langle \rho^i \rangle| \geq 1009$ and in particular $|G| \geq 1009$. Therefore, $|G| =$ 2018 or 4036. If $|G| = 4036$, then $G = D_{4036}$ which contains a rotation of order 2 and is not possible, hence $|G| = 2018$. [**8 marks**]

Part B – Vector Spaces

3. (a) The characteristic polynomial of T is defined to be $\det(xI - T)$.
We say $v \in V$ is an eigenvector if $v \neq 0$ and $Tv = \lambda v$ for some $\lambda \in \mathbb{R}$; then λ is the eigenvalue of v. **[3 marks]**

(b) Standard methods, take T of each of the basis vectors: $T(1) = 1$; $T(x) = 2 - x$; $T(x^2) = x^2 - 1$ and $T(x^3) = 1 - 9x + 6x^2 - x^3$. Hence:

$$[T]_B = \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -1 & 0 & -9 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

The eigenvalues for this matrix are 1 and -1 each with an algebraic multiplicity of 2. It remains to determine whether each eigenvalue has a geometric multiplicity of 1 or 2. Using standard methods, each eigenvalue only has a single eigenvector up to scaling (e.g. count ranks of $T \pm I$, or solve the equations explicitly), and hence has geometric multiplicity 1. The corresponding Jordan canonical form is thus:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[9 marks]

(c) Let $X = [S]_B$, so that:

$$X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This matrix also has 2 eigenvalues each with algebraic multiplicities of 2. However the geometric multiplicity of -1 for S equals the rank of $X + I$ which is 2 (consider column ranks for this to be obvious). Hence there cannot be a basis C such that $[S]_C = [T]_B$, because if there were then $[S]_B$ and $[T]_B$ would be similar (use a change of basis matrix) and thus the geometric multiplicities of -1 would be the same, which they are not. **[8 marks]**

4. (a) Say $\lambda \in k$ (the ground field) and $a \in \mathbb{Z}_{\geq 1}$. Define the $a \times a$ matrix $J_a(\lambda)$ to be the $a \times a$ matrix

$$J_n(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & 0 & \cdots & 0 \\ & & \cdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & 0 & \cdots & \lambda \end{pmatrix}.$$

Define the block direct sum of matrices A and B to be the matrix

$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

with the zeros denoting large rectangles full of zeros. Iterate this construction to get the block diagonal sum of finitely many matrices. **[3 marks]**

(b) Let A be a general 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where here $a + d = t$. Then its characteristic polynomial is given by (standard methods):

$$\begin{aligned} \det(xI - A) &= (x - a)(x - d) - bc \\ &= x^2 - (a + d)x + ad - bc \\ &= x^2 - tx + \det(A). \end{aligned}$$

[3 marks]

(c) Answer is 21. Deal with each part separately, the x^5 implies 5 eigenvalues equal to zero. The block sizes can be split into a variety of ways adding up to five (out of laziness I will write J_n for $J_n(0)$): J_5, $J_4 \oplus J_1$, $J_3 \oplus J_2$, $J_3 \oplus J_1 \oplus J_1$, $J_2 \oplus J_2 \oplus J_1$, $J_2 \oplus J_1 \oplus J_1 \oplus J_1$ or $J_1 \oplus J_1 \oplus J_1 \oplus J_1 \oplus J_1$, so 7 possibilities. For $(x - 2)^3$, there is: $J_3(2)$, $J_2(2) \oplus J_1(2)$ or $J_1(2) \oplus J_1(2) \oplus J_1(2)$, so 3 possibilities. The total possible Jordan canonical forms is simply the product of the two. **[6 marks]**

(d) WLOG A is in Jordan Canonical Form (similar matrices have the same rank), so it's some $J_b(0)$'s plus some $J_c(2)$'s. WLOG all the 0 blocks are before all the 2 blocks. Write $A = B \oplus C$ with B the block sum of the blocks with eigenvalue zero and C the sum of the blocks with eigenvalue 2; then B is 5×5 and C is 3×3.

Ranks of block direct sums add in the obvious way, and the rank of $J_a(\lambda) - \mu.I$ is always a if $\lambda \neq \mu$ because this matrix is invertible (its determinant is non-zero). So the rank of $J_b(2)$ is always b, and so rank of C is 3 and hence C contributes a total of 3 to the rank of A (and A^2, A^3). This means that the rank of B is 3 and the rank of B^2 is 1. Now B is a 5×5 matrix (the algebraic multiplicity of zero) and its rank is $3 = 5 - 2$ so it must be composed of two Jordan blocks by a result in lectures. It can't be $J_4(0) \oplus J_1(0)$ because the square of this has rank 2, so it must be $J_3(0) \oplus J_2(0)$ (up to reordering).

Similarly the rank of $C - 2I$ must be $6 - 5 = 1 = 3 - 2$ so again there are two blocks, so $C = J_1(2) \oplus J_2(2)$ (up to reordering).

The final matrix is then: $J_3(0) \oplus J_2(0) \oplus J_2(2) \oplus J_1(2)$. **[8 marks]**