Course:	M2PM2
Setter:	Setter
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External:	N/A
Date:	January 8, 2015

BSc and MSci MOCK EXAMINATIONS (MATHEMATICS)

January 2015

M2PM2

Algebra II

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BSc and MSci MOCK EXAMINATIONS (MATHEMATICS)

January 2015

This paper is also taken for the relevant examination for the Associateship of the Royal College of Science.

M2PM2

Algebra II

Date: Xday, xth January 2015 Time: 10 am – 12 noon

Credit will be given for all questions attempted but extra credit will be given for complete or nearly complete answers.

Answer **TWO** questions in total – one from Section A (Group Theory) and one from Section B (Vector Spaces).

Calculators may not be used.

Section A - Group Theory

Attempt EITHER Question 1 OR Question 2. You may assume any standard results from the course, unless you are explicitly asked to prove them.

1. (a) Let G and H be groups. Give the definition of a *homomorphism* ϕ from G to H, and the *kernel* of this homomorphism.

   (b) Prove that the kernel $\ker(\phi)$ is a subgroup of G. In addition, prove that $\ker(\phi)$ is a normal subgroup of G.

   (c) Let $\mathbb{R}^*$ denote the group of non-zero real numbers, with the group law being multiplication (you do not need to show that this is a group). Show that the map $\phi : \mathbb{R}^* \times \mathbb{R}^* \to \mathbb{R}^*$ given by

   $$\phi(a, b) = \frac{a}{b}$$

   for $(a, b) \in \mathbb{R}^* \times \mathbb{R}^*$, is a group homomorphism.

   (d) Show that the kernel $\ker(\phi)$ is isomorphic to $\mathbb{R}^*$.

   (e) Show that:

   $$\frac{\mathbb{R}^* \times \mathbb{R}^*}{\ker(\phi)} \cong \mathbb{R}^*.$$

   (f) Find a normal subgroup N of $\mathbb{R}^* \times \mathbb{R}^*$ not equal to $\ker(\phi)$ such that:

   $$\frac{\mathbb{R}^* \times \mathbb{R}^*}{N} \cong \mathbb{R}^*$$

2. The 2018 World Cup is on the way, and Prof. Buzzard cannot help but notice that 2018 is twice a prime number (and you may assume in this question that 1009 is a prime number). As big fan of both football and prime numbers, he is lobbying to get a stadium the shape of a regular polygon with 2018 sides, and is interested in the symmetries of such a stadium. Note: here, D_{2n} is the dihedral group of rotations and reflections for a regular polygon of size n.

   (a) Give the definition of an abelian group. List all the abelian groups (up to isomorphism) of size 4036, and prove that no two groups on your list are isomorphic. You may assume the classification theorem for finite abelian groups.

   (b) Prove or disprove the following statement: $D_{4036} \cong D_{2018} \times C_2$.

   (c) How many cyclic subgroups does D_{4036} have?

   (d) Let G be a subgroup of D_{4036} containing two distinct reflections and not containing the rotation of order 2. Find $|G|$.

Part B - Vector Spaces

Attempt EITHER Question 3 OR Question 4.

3. (a) Let V be a finite dimensional vector space over $\mathbb{R}$. Let $T{:}V \to V$ be a linear transformation. Define:

   - The characteristic polynomial of T;
   - An eigenvalue of T;
   - An eigenvector of T.

   (b) Let V be the vector space consititing of all polynomials over $\mathbb{R}$ of degree at most 3, and B be the basis $1, x, x^2, x^3$ of V. Define $T : V \to V$ by

$$T(p(x)) = p(1-x) + p'(x) - p''(x) \quad \forall p(x) \in V$$

   Find the matrix $[T]_B$ of T with respect to the basis B.
   Find, with proof, a matrix in Jordan canonical form that is similar to $[T]_B$.

   (c) Let V as previous part and define $S : V \to V$ by $S(a + bx + cx^2 + dx^3) = a + (a+b)x + (b-c)x^2 - dx^3$.
   Determine whether or not if there exists a basis C such that matrices $[S]_C$ and $[T]_B$ are equal. Justify your answer.

4. (a) Explain briefly what is meant by a Jordan block $J_a(\lambda)$ and how to take the "block direct sum" of such matrices to make a matrix in Jordan canonical form.

   (b) Prove that if A is a 2×2 matrix with trace t and determinant Δ, then its characteristic polynomial is $X^2 - tX + \Delta$.

   (c) Calculate the total number of non-similar Jordan canonical forms having characteristic polynomial $x^5(x-2)^3$. Give your reasoning.

   (d) Say A is a matrix with characteristic polynomial equal to $x^5(x-2)^3$, and also the following properties:

   - The rank $A - 2I$ is 6;
   - The rank of A is 6;
   - The rank of A^2 is 4;
   - The rank of A^3 is 3.

   Find the Jordan Canonical Form of the matrix A.