Course: M345P11 Galois Theory, Progress Test 2, 3/12/2013.

This test has two questions. They are worth the same number of marks.

In this test you can use any of the theorems you like from the course, unless you are explicitly asked to prove them.

Q1.

(a) Define what it means for a field extension F of E to be a *splitting field* for a (not necessarily irreducible) polynomial $f(x) \in E[x]$.

(b) Say F is a splitting field for a polynomial $f(x) \in E[x]$ of degree d (again not necessarily irreducible). Is it always true that $[F : E] \geq d$? Is it always true that $[F : E] \leq d$? Proofs or counterexamples required!

(c) Define what it means for a finite field extension F/E to be *normal*.

(d) Prove that if F/E is finite and normal, then it is a splitting field.

(e) For each of the polynomials below, compute (with proof, or at least a sketch of a proof) the degree of their splitting fields over $\mathbf{Q}$.

(i) $x^3 - 7$

(ii) $x^4 - 2$

(iii) $x^4 + 4$

(iv) $(x^2 - 5)(x^2 - 7)$

Q2. In this question you can use without proof any result from the course, unless you are explicitly asked to prove it.

(i) Let F/E be an algebraic field extension. What does it mean for $\alpha \in F$ to be *separable* over E? What does it mean for the extension F/E to be *separable*? Give an example (no proof required) of an algebraic extension F/E that is not separable.

(ii) Say $E \subseteq K \subseteq F$ are fields, with F/E algebraic. Prove that F/K and K/E are algebraic. Now assume F/E is separable. Prove that F/K and K/E are separable.

(iii) What does it mean for a finite extension F/E to be *Galois*?

(iv) Say $E \subseteq K \subseteq F$ are field extensions, and F/E is finite and Galois. Is K/E necessarily finite and Galois? Proof or counterexample required.

(v) Say $f(x) \in \mathbf{Q}[x]$ and let F be the splitting field of $f(x)$ over $\mathbf{Q}$. Prove (assuming any results you need from the course) that $F/\mathbf{Q}$ is finite and Galois.