Course: M345P11 Galois Theory, Progress Test 1, 4/11/2013.

This test has two questions, Q1 and Q2. They are worth the same number of marks.

Q1.

(a) Say $f \in \mathbf{Z}[x]$ has degree d, and factors as $f = gh$ with $g, h \in \mathbf{Q}[x]$ of degrees $a, b > 0$. Prove that $f = g'h'$ with $g', h' \in \mathbf{Z}[x]$ of degrees $a, b > 0$.

(b) State Eisenstein's criterion for irreducibility (in $\mathbf{Q}[x]$) of a polynomial $p \in \mathbf{Z}[x]$. No proof required.

(c) For each of the following polynomials $p(x) \in k[x]$, k a field, either prove that $p(x)$ is irreducible in $k[x]$ or give a non-trivial factor of $p(x)$.

(i) $k = \mathbf{Q}$, $p(x) = x^8 - 17$

(ii) $k = \mathbf{R}$, $p(x) = x^8 - 17$

(iii) $k = \mathbf{Z}/2\mathbf{Z}$, $p(x) = x^4 + x^2 + x + 1$

(iv) $k = \mathbf{Z}/2\mathbf{Z}$, $p(x) = x^3 + x + 1$

(v) $k = \mathbf{Z}/3\mathbf{Z}$, $p(x) = x^6 + x^3 + 2$

Q2. In this question you can use without proof any result from the course.

(a) Let $E \subseteq F$ be fields. Define the *degree* $[F : E]$ of the extension. Give an example (with brief justification) of an extension of fields which has infinite degree.

(b) State (without proof) the tower law for degrees of field extensions.

(c) Say $E \subseteq F$ are fields, and $[F : E]$ is finite. If $a \in F$, prove that a is algebraic over E. If d denotes the degree of the minimum polynomial of a over E, prove that d divides $[F : E]$.

(c) Explain clearly how to compute $[\mathbf{Q}(\sqrt{5}, \sqrt{11}) : \mathbf{Q}]$. You may assume $\sqrt{11} \notin \mathbf{Q}(\sqrt{5})$.

(d) Say $\mathbf{Q} \subseteq E \subseteq F$ are fields, and $[F : E] = 2$. Prove that $F = E(a)$ for some $a \in F$ with $a^2 \in E$.

(e) Say $E = \mathbf{Z}/2\mathbf{Z}$ and $E \subseteq F$, where F is a field and $[F : E] = 2$. Prove that there is no $a \in F$ such that $a^2 \in E$ and $F = E(a)$.