Course: M345P11 Galois Theory, Solutions to progress Test 2, 3/12/2013.

Q1.

(a) F is a splitting field for $f(x)$ if $f(x)$ factors as into linear factors ($f(x) = c\prod_{i=1}^{d}(x-\alpha_i)$) and if furthermore $F = E(\alpha_1, \ldots, \alpha_d)$. One mark.

(b) Neither of these are true. For example the splitting field of x^{100} over $\mathbf{Q}$ is just $\mathbf{Q}$ again, of degree $1 < 100$, and the splitting field of $x^3 - 2$ has degree 6 over $\mathbf{Q}$, because adjoining one root gives an extension of degree 3, and thinking of everything as living in the complexes we see that if we adjoin the real cube root of 2 then the other two roots cannot be in the field we obtain, so the splitting field must have degree at least 6 (and in fact has degree exactly 6). One mark for each.

(c) F/E is normal if every irreducible $f(x) \in E[x]$ which is irreducible in $E[x]$ and has a root in F, factors into linear factors in F. One mark.

(d) Say $\alpha_1, \alpha_2, \ldots, \alpha_n$ are an E-basis for F. Let $p_i(x)$ be the minimum polynomial of α_i over E; then $p_i(x)$ has a root in F and hence has all its roots in F by normality. Hence if $p(x) = \prod_i p_i(x)$ then $p(x)$ splits completely and the α_i are amongst its roots, so $p(x)$ is a splitting field for the α_i (because the α_i already generate F over E). Two marks.

(e) (i) If the roots are α, β, γ, then $\alpha/\beta = \omega$ is a non-trivial cube root of unity. One checks that $\mathbf{Q}(\alpha, \beta, \gamma) = \mathbf{Q}(\alpha, \omega)$ because $\beta = \alpha\omega$ and $\gamma = \alpha\omega^2$. Now $[\mathbf{Q}(\alpha) : \mathbf{Q}] = 3$ and $[\mathbf{Q}(\alpha, \omega) : \mathbf{Q}(\alpha)] = 2$ (as ω satisfies a degree two polynomial over $\mathbf{Q}(\alpha)$ but is not in $\mathbf{Q}(\alpha)$, either for degree reasons or an argument involving the reals and complexes). So by the tower law, the splitting field has degree 6 over $\mathbf{Q}$. One measly mark.

(ii) Arguing as above, if α is a 4th root of 2 in the complexes then the four roots are $i^n\alpha$ for $0 \leq n \leq 3$, and the splitting field is $\mathbf{Q}(\alpha, i)$ which by the tower law has degree 8 over $\mathbf{Q}$. One mark.

(iii) The 4th roots of -4 are $\pm 1 \pm i$ so the splitting field is $\mathbf{Q}(i)$, which has degree 2 over $\mathbf{Q}$. One mark.

(iv) The splitting field is $\mathbf{Q}(\sqrt{5}, \sqrt{7})$ and because $\sqrt{7} \notin \mathbf{Q}(\sqrt{5})$ the extension has degree 4 over $\mathbf{Q}$ by the tower law. One mark.

Q2.

(i) $\alpha \in F$ is separable over E if its min poly over E has distinct roots in a splitting field. An algebraic extension F/E is separable if all elements of F are separable over E. One mark each. An example of an algebraic extension which is not separable is $k(t)/k(t^p)$ where k is a field of characteristic p and $k(t)$ denotes the field of fractions of the polynomial ring $k[t]$, which explicitly is the set $\{f(t)/g(t) \ : \ f, g \in k[t], \ g \neq 0\}$. One mark for that too.

(ii) If $\alpha \in F$ is a root of a non-zero polynomial with coefficients in E then the same polynomial has coefficients in K, so F/K is clearly algebraic, and K/E being algebraic is completely obvious. One mark in total for these. Separability – K/E being separable is obvious, and F/K is separable because the min poly of $\alpha \in F$ over K divides the min poly of α over E and hence has distinct roots. One mark for these too (so two in total for the question).

(iii) It means it's normal and separable. One mark.

(iv) It's not true! Normality may fail. For example $\mathbf{Q}(2^{1/3})$ is not normal over $\mathbf{Q}$, but it is contained in the splitting field of $x^3 - 2$, which is Galois over $\mathbf{Q}$ by the next part. Two marks.

(v) The extension is finite because $F = \mathbf{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is finite over $\mathbf{Q}$ (adding one root gives a finite extension, and now use the tower law), it's normal because it's a splitting field (see Q1), and it's separable because inseparability can only occur in characteristic p. Two marks.