A finite-dimensionality result.

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The set-up is this. Let F be a field, $H \subseteq \mathrm{Aut}(F)$ a finite group of automorphisms of F, and let K be the fixed field (that is, $K = \{\alpha \in F : h(\alpha) = \alpha \forall h \in H\}$).

Lemma. *$[F : K] \leq |H|$ (and in particular $[F : K] < \infty$).*

Proof. It suffices to show that if $\alpha_1, \alpha_2, \ldots, \alpha_m \in F$ with $m > |H|$, then there is a non-trivial K-linear relation amongst the α_i. We introduce an auxiliary object $V = \mathrm{Hom}(H, F)$, the maps (of sets) from H to F. This is naturally a vector space over F, of dimension $|H|$. Define $\phi_i \in V$ by $\phi_i(h) = h(\alpha_i)$. Because i runs from 1 to $m > |H|$, there must be a non-trivial F-linear relation between the ϕ_i in V. Choose r as small as possible such that there are $\lambda_i \in F$, not all zero, such that

$$\sum_{i=1}^{r} \lambda_i \phi_i = 0.$$

Clearly $\lambda_r \neq 0$ (or we can just replace r by $r - 1$) and, dividing by λ_r we may assume $\lambda_r = 1$. The displayed equation above means that for all $h \in H$ we have

$$\sum_{i=1}^{r} \lambda_i \phi_i(h) = 0$$

or equivalently, for all $h \in H$,

$$\sum_{i=1}^{r} \lambda_i h(\alpha_i) = 0. \quad (*)$$

Now say $h_1, h_2 \in H$. Setting $h = h_1^{-1} h_2$ in $(*)$ we deduce that

$$\sum_{i=1}^{r} \lambda_i h_1^{-1}(h_2(\alpha_i)) = 0. \quad (**)$$

Hitting $(**)$ with h_1 we deduce

$$\sum_{i=1}^{r} h_1(\lambda_i)(h_2(\alpha_i)) = 0. \quad (***)$$

Now setting $h = h_2$ in $(*)$ and subtracting from $(***)$ we see

$$\sum_{i=1}^{r} (h_1(\lambda_i) - \lambda_i)(h_2(\alpha_i)) = 0. \quad (***)$$

This is true for all $h_2 \in H$, and we deduce

$$\sum_{i=1}^{r} (h_1(\lambda_i) - \lambda_i)\phi_i = 0.$$

But $h_1(\lambda_r) - \lambda_r = 1 - 1 = 0$, and we deduce

$$\sum_{i=1}^{r-1} (h_1(\lambda_i) - \lambda_i)\phi_i = 0.$$

But r was chosen as small as possible amongst the non-trivial relations, so this relation must be trivial. We deduce

$$h_1(\lambda_i) = \lambda_i$$

for all i. But $h_1 \in H$ was arbitrary, and we conclude $\lambda_i \in K$ for all i. Finally setting $h = 1$ in $(*)$ we get

$$\sum_{i=1}^{r} \lambda_i \alpha_i = 0$$

which is the linear relation we seek. □