KMB, November 21, 2013

M3P11 Galois Theory, Problem Sheet 5

1. Let E be a field, and let $D : E[x] \to E[x]$ be as in lectures. If $\alpha \in E$ then prove $D((x-\alpha)^n) = n(x-\alpha)^{n-1}$.

2. Say $E \subseteq F \subseteq K$, with F/E algebraic, and K an algebraic closure of F. Show that K is an algebraic closure of E.

3. Say $E \subseteq F \subseteq K$ are finite extensions. In lectures we showed K/E separable implies K/F and F/E separable. Prove the converse: if K/F and F/E are separable then so is K/E. Hint: don't try to do it directly from the definition (by which I mean that I don't know how to do it directly from the definition).

4. In this (long and skippable) question I will explain how to build a finite extension F/E which is not simple (i.e. not of the form $F = E(\alpha)$) and which has infinitely many intermediate subfields. As we will see/have seen in the course, the moment things are separable this can't happen, so this example must involve infinite fields of characteristic p. Such weird field extensions are really beyond the scope of the course, so if you want to skip this question then feel free.

Let k be a field. Consider the field $k(s,t)$ of "rational functions in s and t"; an element of $k(s,t)$ is a ratio $f(s,t)/g(s,t)$ of polynomials $f, g \in k[s,t]$ in two variables, with $g(s,t)$ assumed non-zero, and of course we define $f_1/g_1 = f_2/g_2$ iff $f_1g_2 = f_2g_1$. Pedants will want me to say that $k(s,t)$ is equivalence classes of pairs (f,g) with $f, g \in k[s,t]$ and $g \neq 0$, under the equivalence relation $(f_1, g_1) \sim (f_2, g_2)$ iff $f_1g_2 = f_2g_1$. Algebra whizzes will realise that $k(s,t)$ is just the field of fractions of $k[s,t]$.

(i) Convince yourself that $k(s,t)$ is a field. Convince yourself that the notation is sensible too – any subfield of $k(s,t)$ containing k, s and t must be all of $k(s,t)$.

(ii) Now let k be an infinite field of characteristic p (for example $k = (\mathbb{Z}/p\mathbb{Z})(r)$ with r an abstract variable), and set $F = k(s,t)$ and let E be the subfield $k(s^p, t^p)$ of ratios of polynomials in s^p and t^p. Show that $[F : E] = p^2$ (and in particular the extension is finite).

(iii) If $\lambda \in F$ then show that $\lambda^p \in E$. Deduce that $F \neq E(\lambda)$ for any $\lambda \in F$.

(iv) If $\gamma \in k$ then let E_γ be the field $E(s + \gamma t)$. Prove that for $\gamma, \delta \in k$ we have $E_\gamma = E_\delta$ iff $\gamma = \delta$. Deduce that F/E has infinitely many subfields.

5. Let E be any field of characteristic not equal to 2, say $e \in E$ is an element and that there is no $d \in E$ with $d^2 = e$. Let F be the splitting field of $x^2 - e$ over E. What is $[F : E]$? Prove F/E is finite and Galois. What is $\mathrm{Gal}(F/E)$?

6. Say $E = \mathbb{Q}$, and let F be the splitting field of $x^4 - p$ where p is a prime number. What is $[F : E]$? What is $\mathrm{Gal}(F/E)$?

7. Say $E = \mathbb{Q}$ and let F be the splitting field of $x^p - 1$, where p is a prime number. What is $[F : E]$? What is $\mathrm{Gal}(F/E)$?