KMB, November 15, 2013

M3P11 Galois Theory, Problem Sheet 4

[NB this is v3: I got the definition of normal closure wrong on v2 :-( ]

1. Prove that any ring homomorphism between fields is injective. Recall that fields have a 0 and a 1, with $0 \neq 1$, and a ring homomorphism preserves $+$, $\times$, 0 and 1.

2. Say $E \subseteq K$ are fields, and that K is algebraically closed. If F denotes the subset of K consisting of elements of K which are algebraic over E, show that F is also an algebraically closed field, which is furthermore algebraic over E.

[In fact, up to isomorphism, there is only one extension of E which is algebraically closed and algebraic over E; such a field is called an *algebraic closure* of E.]

3. Establish (with proofs) whether the following extensions of $\mathbb{Q}$ are normal or not.

(i) $\mathbb{Q}(\sqrt{6})$

(ii) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$

(iii) $\mathbb{Q}(7^{1/3})$

(iv) $\mathbb{Q}(7^{1/3}, e^{2\pi i/3})$

(v) $\mathbb{Q}(\sqrt{1+\sqrt{7}})$ [NB in the first version of this example sheet, which was up for a few hours on Monday, I had $\mathbb{Q}(\sqrt{1+\sqrt{2}})$, which seems to me to be a fair bit trickier].

(vi) $\mathbb{Q}(\sqrt{2+\sqrt{2}})$

4. Prove that if $E \subset F$ and $[F : E] = 2$ then F/E is normal.

5. Say $E \subseteq F$ is an algebraic field extension. An extension $F \subseteq K$ is called a *normal closure* of F/E if K/E is normal, and furthermore if $F \subseteq M \subseteq K$ and M/E is normal, then $M = K$. Prove that every finite field extension has a normal closure.

6. Say $E \subseteq F$ is an extension of fields with $[F : E]$ finite, and M, N are both subfields of F containing E. Assume that M/E and N/E are both normal.

(i) Prove that $(M \cap N)/E$ is normal.

(ii) Prove that MN (defined as the smallest subfield of F containing both M and N) is normal over E as well.

7. Prove that if $E \subseteq F \subseteq K$ are fields, if F/E is finite and normal, and if $i : F \to K$ is any field homomorphism which is the identity on E, then $i(F) = F$. Show that this might not be the case if F/E is not normal.