KMB, 21/10/13

M3P11 Galois Theory, Problem Sheet 2

1. (a) Prove that if $n \in \mathbb{Z}$ and $\sqrt{n} \notin \mathbb{Z}$ then $\sqrt{n} \notin \mathbb{Q}$.

(b) Prove that $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$. What is the minimum polynomial of $\sqrt{3}$ over $\mathbb{Q}(\sqrt{2})$?

(c) Use the Tower Law to prove that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$.

2. (a) Prove that $\mathbb{Q}(\sqrt{2}, \sqrt{3}) = \mathbb{Q}(\sqrt{2} + \sqrt{3})$. Hint: the smallest subfield of the complex numbers containing $\mathbb{Q}$ and $\sqrt{2} + \sqrt{3}$ must contain loads of other things too: write some of them down.

(b) Deduce that $x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$.

3. Is $\sqrt{10} \in \mathbb{Q}(\sqrt{6}, \sqrt{15})$?

4. In this question, if $\alpha \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{\geq 1}$ then by $\alpha^{1/n}$ or $\sqrt[n]{\alpha}$ I mean the unique positive real number β with $\beta^n = \alpha$. (This removes ambiguities about a general complex number having n complex roots in this question).

(i) Set $\gamma = (1 + \sqrt{3})^{1/3}$. Prove that γ is algebraic. What is its degree over $\mathbb{Q}$? What is its degree over $\mathbb{Q}(\sqrt{3})$?

(ii) Set $\delta = (10 + 6\sqrt{3})^{1/3}$. Prove that δ is algebraic. What is its degree over $\mathbb{Q}$? What is its degree over $\mathbb{Q}(\sqrt{2})$?

5. Prove that any subfield of $\mathbb{C}$ must contain $\mathbb{Q}$.

6. Prove the part of the tower law that I didn't check: if $E \subseteq K \subseteq F$ are fields, and one of $[K : E]$ or $[F : K]$ is infinite, then $[F : E]$ is infinite.

Remark: for those unfamiliar with infinite-dimensional vector spaces, a vector space V over a field is infinite-dimensional iff it has no finite spanning set, iff for every $n \geq 1$ there exist n elements $v_1, v_2, \ldots, v_n$ which are linearly independent.

7. Let A denote the set of complex numbers which are algebraic (over $\mathbb{Q}$).

(i) Prove that A is a field.

(ii) Prove that $[A : \mathbb{Q}] = \infty$. Hint: you can use Eisenstein to construct irreducible polynomials of large degree...

(iii) Prove that A is a countable set.

(iv) Prove that $[\mathbb{C} : A] = \infty$.