M2PM2 Algebra II, Progress test 1, 15/10/2013, solutions.

Q1.

i. This is false – a counterexample is the dihedral group D_{12}, the symmetries of a regular hexagon. If it were cyclic then it would have to have an element of order 12. But what could this element be? The subgroup of rotations has order 6, so by Lagrange every rotation has order dividing 6 and hence not equal to 12; the other six elements are reflections, but all of these have order 2. So there are no elements of order 12. Two marks (and perhaps lose one mark for just saying "D_{12} is obviously not cyclic" because we deserve a little more than that surely).

ii. This is not true: the cyclic group C_7 has order 7, but it has no subgroup of order 6 by Lagrange. Two marks (but I think that just saying "false by Lagrange" is not quite enough because you have to rule out the possibility that there are no groups of order 7 at all!)

iii. This is true: just let G be, say, the cyclic group C_{1000} and let g be a generator; then $g \neq e$ (e the identity) (because g is a generator), and $g^{1000} = e$ but $g^{1001} = g \neq e$. Two easy marks.

iv. This is not true and I asked this question on the first example sheet, so those who prepared properly should have seen this one coming. A counterexample would be $G = S_3$, the symmetric group. Apart from the identity, all the elements are 2-cycles (of order 2) and 3-cycles (of order 3). There are lots of dihedral counterexamples too, although these are slightly trickier to spot. Two marks for this.

v. This isn't true at all (anyone who thinks it is should think harder about how cyclic groups work, e.g. rotate a beer-mat a few times). If G is the cyclic group of order 3 (or indeed any odd prime) and $a \neq b$ are non-identity elements of G, then $\langle a \rangle$ and $\langle b \rangle$ can't be $\{e\}$ so must be G by Lagrange's theorem. Alternatively you can just give a direct calculation rather than Lagrange. Two marks.

Q2.

i. Six rotations and six reflections. One mark for this.

ii. Any subgroup of order 2 must contain the identity plus an element which must have order 2 (by Lagrange, if you like). So we just need to count the number of elements of order 2. Every reflection has order 2, so that's six, and if ρ is the rotation by $2\pi/6$ then the rotations are $\{e, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$. Squaring these gives $e, \rho^2, \rho^4, e, \rho^2, \rho^4$, so ρ^3 has order 2 and none of the others do. So there are seven subgroups of order 2, corresponding to the seven elements of order 2. Three marks for this.

iii. It's a standard fact that $\sigma\rho = \rho^{-1}\sigma$, so (multiply both sides by σ on the right and use $\sigma^2 = e$) we have $\sigma\rho\sigma = \rho^{-1}$. So the result is true for $n = 1$. For general $n \geq 2$ we use induction: if $\sigma\rho^{n-1}\sigma = \rho^{1-n}$ then multiplying the left hand side by $\sigma\rho\sigma$ and the right hand side by ρ^{-1} (which equals $\sigma\rho\sigma$) we deduce $\sigma\rho^{n-1}\sigma^2\rho\sigma = \rho^{-n}$, and the left hand side simplifies to $\sigma\rho^n\sigma$, so we're done. Alternatively just note that $(\sigma\rho\sigma)^n = \sigma\rho^n\sigma$ because there's lots of cancellation. Three marks for this.

iv. There is! If we write $D_{12} = \{e, \rho, \ldots, \rho^5, \sigma, \rho\sigma, \ldots, \rho^5\sigma\}$ as usual then $z = \rho^3$ works. For if g is a rotation, then $g = \rho^n$ for some n, and $zg = \rho^{n+3} = gz$. On the other hand, if

g is a reflection, then $g = \rho^n\sigma$ for some n, where σ is a fixed reflection, and $zg = \rho^{n+3}\sigma$, whereas (using the standard fact that $\sigma\rho = \rho^{-1}\sigma$)

$$\begin{aligned} gz &= \rho^n\sigma\rho^3 \\ &= \rho^n\sigma\rho^3\sigma\sigma \\ &= \rho^n\rho^{-3}\sigma \text{ by the previous part} \\ &= \rho^{n-3}\sigma \\ &= \rho^{n-3}\rho^6\sigma \text{ because } \rho \text{ has order } 6 \\ &= \rho^{n+3}\sigma \end{aligned}$$

and hence $zg = gz$. Three marks for this tricky question.