M2PM2, test 2, 19/11/13.

This test consists of two questions. Both questions carry the same amount of marks.

Q1. In this question you can assume any results you like from the course.

i. How many elements are there in the group $C_2 \times C_4 \times C_{12}$? How many have order dividing 6?

ii. $\mathbf{Z}$ is a subgroup of $\mathbf{R}$ (you may assume this). But is it a normal subgroup of $\mathbf{R}$?

iii. Let M be a subgroup of the group G, and let N be a subgroup of the group H. Explain briefly why $M \times N$ is a subgroup of $G \times H$. If M is a normal subgroup of G, and N is a normal subgroup of H, is $M \times N$ a normal subgroup of $G \times H$? (Proof or counterexample needed.)

iv. Proof or counterexample: if G and H are groups, and G and H are isomorphic, then there is only one isomorphism $\phi : G \to H$.

v. Let $\sigma \in S_{2013}$ be the permutation of $\{1, 2, 3, \ldots, 2013\}$ sending x to $2014 - x$. What is the signature of this permutation?

Q2. In this question you can assume any results you like from the course, apart from when you're explicitly asked to prove them (i.e. parts (ii) and (iii)).

i. Let $A = (a_{ij})$ be an $n \times n$ matrix. Write down the definition of the *determinant* of A.

ii. Prove (directly from the definition) that if A is upper triangular, then $\det(A) = a_{11}a_{22}\ldots a_{nn}$.

iii. Prove (directly from the definition) that $\det(A^T) = \det(A)$ (here A^T denotes the transpose of A).

iv. Proof or counterexample: If A and B are $n \times n$ matrices with real coefficients, then $\det(A + B) = \det(A) + \det(B)$.

v. What is the determinant of the following matrix: $\begin{pmatrix} M & 2 & P & M & 2 \\ 1 & Z & L & 3 & 3 \\ T & 4 & N & D & I \\ H & E & A & R & T \\ M & 2 & P & M & 2 \end{pmatrix}$?

vi. Let A be the 100×100 matrix $\begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$.

In other words $A = (a_{ij})$ with $a_{ij} = 1$ if $i = j + 1$ or $i = j - 1$, and $a_{ij} = 0$ otherwise. What is the determinant of A?