KMB, 15th November 2013

M2PM2 Algebra II, Solutions to Problem Sheet 7

1. (a) The only non-zero term in the sum defining the determinant is the one mentioning $a_{13}a_{24}a_{32}a_{41}a_{55}$, which corresponds to the 4-cycle $\pi = (1324)$, which has signature -1. Hence the determinant is -1.

More generally, a *permutation matrix* is a matrix with exactly one "1" in each row and each column, and all other entries are zero. Each such matrix defines a permutation, and the determinant of the matrix is the signature of the permutation. For example the elementary matrices B_{ij} correspond to the permutation $(i\ j)$ and have determinant -1, the signature of a transposition.

(b) This matrix is lower-triangular, so by a result in lectures the determinant is just the product of the diagonal entries, which is -42.

(c) Expanding down the second column, the determinant is

$$\ell\left|\begin{pmatrix} m & 0 & a & b \\ n & e & d & c \\ p & 0 & 0 & k \\ h & 0 & 0 & t \end{pmatrix}\right|$$

and expanding the above 4×4 matrix down the second column gives

$$\ell e\left|\begin{pmatrix} m & a & b \\ p & 0 & k \\ h & 0 & t \end{pmatrix}\right|.$$

Now expanding down the second column of the remaining 3×3 matrix we get

$$-\ell ea\left|\begin{pmatrix} p & k \\ h & t \end{pmatrix}\right|$$

(note the minus sign, that we pick up because we're going down the second column rather than the first), and we can do the 2×2 matrix by hand, giving the solution as $\ell ekha - pate\ell$.

d) This matrix has determinant zero. For if the matrix is (a_{ij}) then we see that $a_{ij} = 0$ if $i \in \{3,4,5\}$ and $j \in \{1,2,3\}$. But thinking about the definition of determinant, if π is in S_5 then $\pi(3)$, $\pi(4)$ and $\pi(5)$ are three distinct elements of $\{1,2,3,4,5\}$, and hence they cannot *all* be in the set $\{4,5\}$, which only has size 2. In particular there must be some $i \in \{3,4,5\}$ with $\pi(i) \in \{1,2,3\}$. Hence this $a_{i\ \pi(i)}$ term will be zero, so the term corresponding to π in the sum defining the determinant must be zero. Hence the determinant is zero! There are also other ways to see this – for example expanding down rows or columns gives it to you without too much trouble.

2. (a) $|A(\alpha)| = \alpha - 1$. The most painless way to see this, I think, is to expand down the third column, and then note that one of the resulting minors has one of its columns equal to the negative of another one, and hence has determinant equal to zero, so this brings us down to a 3×3 matrix, which is a reasonable computation. Note that one nice check to see if you've made a slip: if $\alpha = 1$ then the first and second rows of the matrix coincide so the determinant should be zero, and hence $|A(\alpha)|$ has to be a multiple of $\alpha - 1$.

(b) $\alpha_0 = 1$ (using result from lectures that system $Ax = 0$ has a nonzero solution for x iff $|A| = 0$).

(c) For $\alpha < 1$, $|A(\alpha)| < 0$. If $B^2 = A(\alpha)$ then by the multiplicativity of det, $|B|^2 = |A(\alpha)| < 0$, which is impossible if B is real.

3. Expanding down the first column, we get

$$|A_n| = 2|A_{n-1}| + \left| \begin{pmatrix} -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 & 0 \\ & & & & \dots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \right|$$

and expanding the big matrix above along the first row gives

$$|A_n| = 2|A_{n-1}| - |A_{n-2}|.$$

Now check by hand that $|A_1| = 2$ and $|A_2| = 3$, and then $|A_n| = n+1$ follows via a very easy (strong) induction, because it's true for $n = 1, 2$ and if we believe it for all numbers less than n then we see $|A_n| = 2(n-1+1)-(n-2+1) = 2n-n+1 = n+1$.

4. Expanding down the first column we get $|B_n| = |B_{n-1}| + |B_{n-1}|$, hence $|B_n| = 2|B_{n-1}|$. An easy check gives $|B_1| = 1$ (and $|B_2| = 2$ if you're paranoid), and now $B_n = 2^{n-1}$ follows by an easy induction.

5. Let's prove this by induction on s. If $s = 1$ then the result follows by expanding down the first column. If $s > 1$ and we know the result for $s - 1$ then again we expand down the first column, and deduce

$$|A| = b_{11}|A_{11}| - b_{21}|A_{21}| + \dots + (-1)^{s-1}b_{s1}|A_{s1}|.$$

Here, of course A_{ij} means the (i, j)th minor of A. The trick is to notice that the inductive hypothesis applies to all the A_{i1}, showing that $|A_{i1}| = |B_{i1}| \cdot |D|$, where B_{i1} is the $(i, 1)$th minor of B. Now reconstructing, we get

$$|A| = b_{11}|B_{11}||D| - b_{21}|B_{21}||D| + \dots$$

and this is just $|B|.|D|$ (as can be seen by expanding $|B|$ down the first column).

There is also a fancier direct proof, which goes something like this: consider $\sigma \in S_n$, with $n = s + t$. If there is some $i \geq s + 1$ such that $\sigma(i) \leq s$, then $a_{i,\sigma(i)} = 0$ (as we've just landed in the area where all the zeros are). So the only σ that contribute to the determinant must send $\{s+1, s+2, \dots, s+t\}$ to $\{s+1, s+2, \dots, s+t\}$ and hence must send $\{1, 2, \dots, s\}$ to $\{1, 2, \dots, s\}$; hence $\sigma = \pi_1\pi_2$ with π_1 a permutation of $\{1, 2, \dots, s\}$ and π_2 a permutation of $\{s+1, \dots, s+t\}$; then the σ term in $\det(A)$ corresponds to the product of the π_1 term in $\det(B)$ and the π_2 term in $\det(D)$.

6. (a) Suppose $|A| = 0$. Then A is not invertible (by lectures). It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $ABC = I$, so BC would be the inverse of A, contradiction). Hence $|AB| = 0$, again by lectures.

(b) Similar: suppose $|B| = 0$. Then B is not invertible. It follows that AB is also not invertible (if it were, say the inverse was C, we'd have $CAB = I$, so CA would be the inverse of B, contradiction). Hence $|AB| = 0$.

7. There are lots of ways of doing these rather elementary calculations.

(a) $|A_i(r)| = r$ because $A_i(r)$ is upper-triangular, and hence by lectures its determinant is the product of the diagonal entries, which is $1\times 1\times\cdots\times 1\times r\times 1\times\cdots$ which is r.

$|B_{ij}| = -1$ because B_{ij} is obtained from the identity matrix by swapping the i and jth rows, and switching two rows changes the sign of the determinant by lectures.

$|C_{ij}(r) = 1$ because $C_{ij}(r)$ is either upper triangular or lower triangular, so in either case its determinant is the product of its diagonal entries, all of which are 1.

(b) Easy check: multiplying diagonal matrices is easy: you just multiply the entries pointwise. So $A_i(r)A_i(s) = A_i(rs)$ and in particular $A_i(r)A_i(r^{-1}) = A_i(1) = I$, so $A_i(r^{-1})$ must be the inverse of $A_i(r)$.

Next, $B_{ij}M$ is just the matrix obtained from M by switching the ith and jth rows of M, as can easily be seen by writing down the formula for matrix multiplication. Hence $B_{ij}B_{ij} = I$ the identity matrix, so $B_{ij} = B_{ij}^{-1}$.

Finally, $C_{ij}(r)M$ is the matrix obtained from M by adding r times the jth column to the ith column. If we do this to $C_{ij}(-r)$ then we get the identity matrix. Hence $C_{ij}(r)C_{ij}(-r) = I$.

8. $\begin{pmatrix}1&0&0\\0&1&0\\3&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&2&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&3\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&0\\0&1&0\\0&0&1\end{pmatrix}$ is one answer.

9. First bit was done in lectures. To show $\sim$ an equivalence relation: obviously $A \sim A$; if $A \sim B$ then $B = E_1 \dots E_k A$, hence $A = E_k^{-1} \dots E_1^{-1} B$, so $B \sim A$ as all E_i^{-1} are elementary; and if $A \sim B$ and $B \sim C$, then $B = E_1 \dots E_k A$ and $C = F_1 \dots F_l B$ with all E_i, F_i elementary, so $C = F_1 \dots F_l E_1 \dots E_k A$, hence $A \sim C$.