KMB, 12th November 2013

M2PM2 Algebra II Solutions to Problem Sheet 6

1. (a) G is cyclic, hence abelian, so $N \lhd G$ automatically. If $G = \langle x \rangle$, then every coset in the factor group G/N is of the form $Nx^r = (Nx)^r$, so G/N is generated by Nx, so is cyclic.

(b) The map $x \to [x]$ is a group homomorphism $\mathbb{Z} \to \mathbb{Z}_n$ with kernel $n\mathbb{Z}$. So the first isomorphism theorem says $\mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}_n$.

(c) If $x = (g_1, h_1)$ and $y = (g_2, h_2)$ then $xy = (g_1g_2, h_1h_2)$, so $\phi(xy) = h_1h_2 = \phi(x)\phi(y)$ and ϕ is hence a homomorphism. Its kernel is clearly the elements of $G \times H$ of the form (g, e) with $g \in G$. It is now easy to check that the map $G \to \ker(\phi)$ sending $g \in G$ to (g, e) is an isomorphism.

(d) $G = S_{100} \times C_2 \times C_2 \times C_2$, $N = \{(g, 1, 1, 1) : g \in S_{100}$ (note N is normal and the quotient is what we want, by part (c) and the First Isomorphism Theorem).

2. (i) There are such homomorphisms for $r = 1$ ($\phi(g) = 1$ for all $g \in S_n$) and for $r = 2$ ($\phi(g) = sgn(g)$ for all $g \in S_n$, which is surjective if $n \geq 2$).

We now show there is no such homomorphism if $r > 2$. For suppose $\phi : S_n \to C_r$ is a surjective homomorphism. Then $\phi((i\ j))$ has order 1 or 2, so is ± 1. But *every* permutation is a product of 2-cycles (see lectures), so this forces $\phi(g) = \pm 1$ for all $g \in S_n$. Therefore $r \leq 2$.

(ii) This is hard! Well done if you managed to get anywhere with it.

Suppose $S_n/N \cong C_r$. Then $r = 1$ or 2 by (i). If $r = 1$ then $N = S_n$. Now suppose $r = 2$, so $S_n/N \cong C_2$. Then for any $x \in S_n$, $(Nx)^2 = N$, so $x^2 \in N$. Now we show that every even permutation lies in N, i.e. $A_n \subseteq N$ – hence $A_n = N$ as they have the same size. So let $y \in A_n$. Then y is a product of an even number of 2-cycles: say $y = t_1t_2 \ldots t_{2r-1}t_{2r}$ with each t_i a 2-cycle. If t_1, t_2 are disjoint 2-cycles, say $(ij), (kl)$, then $t_1t_2 = (ij)(kl) = (ikjl)^2$; and if not, then $t_1t_2 = (ij)(jl) = (ijl) = (ilj)^2$. So t_1t_2 is the square of a permutation, hence lies in N. Similarly $t_3t_4 \in N$, and so on, hence $y \in N$. That's it.

3. $G = D_8$: possible H have size dividing 8, so 1,2,4 or 8. Clearly $H = C_1$ and $H = G = D_8$ are possible, and looking at the kernel of $G \to H$ shows that D_8 is the only possibility for H of size 8. We also know that $H = C_2$ is possible, because it is the quotient of G by $\langle \rho \rangle$.

So the only question is which H of size 4 are possible. Q8 of Sheet 5 shows only $C_2 \times C_2$ is possible. So the list of H is C_1, C_2, $C_2 \times C_2$ and D_8.

$G = D_{12}$: by lectures the possible H are the factor groups G/N for $N \lhd G$. The story for $|H| = 1$ or 2 or 12 is clear (C_2 is possible because it's the quotient of G by $\langle \rho \rangle$). So the only question left is which H of size 3, 4 or 6 are possible.

For $|H| = 3$ we must have $H \cong C_3$, which has one element of order 1 and two of order 3. If $\phi : G \to H$ is a surjection with kernel N then $|N| = 4$, but for τ any element of G of order 2 we must have $\phi(\tau)$ of order dividing 2, and hence $\phi(\tau) = 1$ which implies $\tau \in N$. This is a contradiction because N has size 4 but we've just written down at least six elements of it (the six reflections). So $|H| = 3$ does not occur.

For $|H| = 4$ we need $|N| = 3$, and the only normal subgroup of size 3 is $N = \langle \rho^2 \rangle$ (there are only two elements of G having order 3), for which G/N can be checked to be $C_2 \times C_2$. For $|H| = 6$ we need $|N| = 2$, and the only normal subgroup of size 2 is $N = \langle \rho^3 \rangle$ (the reflections don't generate normal subgroups, by sheet 5 Q6c), for which $G/N \cong D_6$. The possible H are hence C_1, C_2, $C_2 \times C_2$, D_6, D_{12}.

$G = S_4$: if H is cyclic, it is C_1 or C_2 by Q2. If not, $|H|$ is not prime and divides 24, so it's 4,6,8,12 or 24. For $|H| = 24$ we have $H = S_4$. For $|H| = 12$ or 8 we have $|N| = 2$ or 3; but one can check S_4 has no normal subgroups of size 2 or 3 (S_4 has lots of subgroups of these sizes, but none are normal). For $|H| = 6$ we have $H \cong C_6$ or D_6; but C_6 is not possible by Q2. However D_6 is possible! The normal subgroup is $V := \{e, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$, which can be checked to be normal because if $e \neq g \in V$ then $x^{-1}gx$ is an even permutation of order 2 and is hence in V again. Finally check that if $x := V(123)$ and $y := V(12)$ then $o(x) = 3$, $o(y) = 2$ and $y^{-1}xy = x^{-1}$ so we have a D_6 here. Well done if you spotted this one.

Finally for $|H| = 4$ we have $|N| = 6$ so $N \cong D_6$. Then N has only two elements of order 3 (say (123) and (132)) and you can check that S_4 has no such normal subgroup by finding $x \in S_4$ such that $x^{-1}(123)x \neq (123)$ and $x^{-1}(123)x \neq (132)$.

Conclude that the possibilities for H are: C_1,C_2,D_6,S_4.

4.

(a) Consider an axis going through the centre of the tetrahedron and one of the corners (and the middle of the opposite side). There are four such axes, one per corner, and each one gives us two rotations (one by $2\pi/3$ and one by $4\pi/3$), so there's 8. The identity is a ninth, and the three rotations by π via each of the three axes that go through two opposite edges give me three more, so that's 12.

(b) Consider a rotation of a tetrahedron. There are four possibilities for the face that lands on the bottom after you've done the rotation, and then three possibilities for the "front" face, and this specifies the rotation completely – hence there are at most 12 rotations, and so, by (a), exactly 12.

(c) Consider the injective group homomorphism from the rotation group to the group of permutations of the vertices. The rotations of order 3 give rise to 3-cycles, and the ones of order 2 give rise to permutations with cycle type that of $(1\ 2)(3\ 4)$. All of these permutations are even, so we have an injection from a group of size 12 to A_4 which must hence be a bijection.

(d) Imagine a reflection through a plane which contains an edge of the tetrahedron and bisects the opposite edge. This is an isometry (although we can't realise it in "the real world") and it permutes two vertices and fixes the other two. So there's an injection from the full symmetry group to S_4 and the image contains A_4 and a transposition, so it's also a surjection and hence an isomorphism.

5.

(a) I can think of the identity, then 9 rotations through axes that go through the midpoints of two opposite faces, 6 rotations through axes that go through the midpoints of two opposite edges, and 8 through axes that go through two opposite corners, giving a total of 24 that I can think of.

(b) Any rotation sends some face to the bottom (and there are six possibilities for this face), and then one of the adjacent faces to the front (and there are four possibilities for this face) giving a total of $6 \times 4 = 24$ possibilities at most.

(c) If you consider each rotation in the symmetry group of the cube as a permutation of the 4 long diagonals, you get the eight 3-cycles by rotating about axes which are long diagonals; you get the six 2-cycles by rotating about axes through the mid-points of diagonally opposite sides; and you get the 4-cycles and $(2,2)$-perms by rotating about axes through the mid points of opposite faces.