KMB, 5th November 2013

M2PM2 Algebra II, Solutions to Problem Sheet 5.

1.

a) Yes – image C_4, kernel C_3.

b) No (does not satisfy $\phi(xy) = \phi(x)\phi(y)$ – for example if $x = (1\ 2\ 3)$ and $y = (2\ 3\ 4)$, then x and y have order 3 but $xy = (2\ 1)(3\ 4)$ has order 2).

c) Yes – image $\mathbb{Z}_n$ (ϕ is surjective), kernel $\langle n \rangle = \{kn : k \in \mathbb{Z}\}$.

d) Yes, image $\mathbb{R}_{>0}$, kernel $\{1\}$ (in fact this ϕ is an isomorphism, with inverse map $x \mapsto x^2$).

e) No (map not even well-defined! For $[0]_6 = [6]_6$ but $[0]_7 \neq [6]_7$.)

f) Yes, image $\{[1], [2], [4]\}$ and kernel $\{[0], [3]\}$, as you can see by simply writing what ϕ does to each element of $\mathbb{Z}_6$.

2.

(a) This is straightforward if you can remember what an equivalence relation is! To prove $x \sim x$ choose $g = e_G$ and then note that $g^{-1}xg = x$. Next, if $x \sim y$, then for some g we have $g^{-1}xg = y$, so if $h = g^{-1}$ then $h^{-1}yh = gyg^{-1} = x$, and so $y \sim x$. Finally, if $g^{-1}xg = y$ and $h^{-1}yh = z$, then $(gh)^{-1}x(gh) = h^{-1}g^{-1}xgh = h^{-1}yh = z$ and hence $\sim$ is transitive. We're done!

(b) Assume first that H is normal. Say $x \sim y$ and $x \in H$. It suffices to prove that $y \in H$. By definition of $\sim$, there exists g such that $y = g^{-1}xg$. Hence $y \in g^{-1}Hg = H$ by normality, and we're done.

Conversely suppose that H is a union of conjugacy classes. We need to check (using 6.4) that if $g \in G$ is arbitrary, then $g^{-1}Hg \subseteq H$. But if $y \in g^{-1}Hg$ then $y = g^{-1}xg$ for some $x \in H$, and $y \sim x$. Now H is a union of conjugacy classes, so $x \in H$ implies $y \in H$, which is what we wanted to show.

3. (a) Note first that if $\sigma = (1\,2\,3\,4\,5)$ then $\sigma^2 = (1\,3\,5\,2\,4)$. We want to find some x such that $x^{-1}\sigma x = \sigma^2$, and multiplying on the left by x it's equivalent to solve $\sigma x = x\sigma^2$. One checks explicitly that if $x = (2\ 4\ 5\ 3)$ then $\sigma x = (2\ 5\ 4\ 1) = x\sigma^2$, so we're done.

(b) Again the question boils down to solving $\sigma x = x\sigma^2$, but this time $x \in A_5$, so our solution $x = (2\ 4\ 5\ 3)$ above does not work, as a 4-cycle has signature -1. Are there any solutions at all? Say x is a solution and $x(1) = n$. Evaluating $\sigma x = x\sigma^2$ at 1, the left hand side is $\sigma(n) = n + 1$ (with the convention that $5 + 1 = 1$, i.e. work mod 5), and the right hand side is $x(3)$. So $x(1) = n$ implies $x(3) = n + 1$. Evaluating at 3 we deduce $x(5) = n + 2$, and then $x(2) = n + 3$ and $x(4) = n + 4$. Now we know $1 \leq n \leq 5$, and trying all possibilities we deduce that $x = (3\ 2\ 4\ 5)$ or $(1\ 2\ 5\ 4)$ or $(1\ 3\ 4\ 2)$ or $(1\ 4\ 3\ 5)$ or $(1\ 5\ 2\ 3)$; these are hence the only solutions to $\sigma x = x\sigma^2$ in S_5, and none of the solutions are in A_5, so σ and σ^2 are not conjugate in A_5.

4. Say $g \in G$ and $x \in M \cap N$. Then $g^{-1}xg \in M$ (as $M \lhd G$) and $g^{-1}xg \in N$ (as $N \lhd G$), hence $g^{-1}xg \in M \cap N$. Thus $g^{-1}(M \cap N)g \subseteq M \cap N$. This is true for all $g \in G$, so Lemma 6.4 implies $M \cap N \lhd G$.

5. It is not hard to check that ϕ is a homomorphism: $\phi(xy) = g^{-1}xyg$ and $\phi(x)\phi(y) = g^{-1}xgg^{-1}yg = g^{-1}xyg$. Furthermore, ϕ is a bijection, because if I define $\psi : G \to G$ by $\phi(y) = gyg^{-1}$ then it is easily checked that ψ is an inverse for ϕ (the gs cancel). Hence ϕ is an isomorphism.

To show that ϕ may not be the identity map, we just need to find a group G and elements x and g such that $g^{-1}xg \neq x$, or equivalently (multiply on the left by g) that $gx \neq xg$. But this is easy: for example take $G = S_3$ and $x = (1\ 2)$ and $g = (1\ 3)$.

6.

(a) Let $x = \rho^{ri} \in \langle \rho^r \rangle$. Then

$$\rho^{-j}x\rho^j = \rho^{-j+ri+j} = x,$$
$$(\rho^j\sigma)^{-1}x(\rho^j\sigma) = (\sigma\rho^{-j})\rho^{ri}(\rho^j\sigma) = \sigma\rho^{ri}\sigma = \rho^{-ri}\sigma\sigma = \rho^{-ri} = x^{-1}.$$

Hence $g^{-1}xg \in \langle \rho^r \rangle$ for all $g \in D_{2n}$, so $\langle \rho^r \rangle \triangleleft D_{2n}$.

(b) $\rho^{-1}(\rho^r\sigma)\rho = \rho^{-1}\rho^r\rho^{-1}\sigma = \rho^{r-2}\sigma \notin \langle \rho^r\sigma \rangle$ (using $n \geq 3$ here), so $\langle \rho^r\sigma \rangle$ is not normal in D_{2n}.

7. (a) Let H be a subgroup of D_{2p}, and assume $H \neq \{e\}$ or D_{2p}. By Lagrange, H has size 2 or p, so H is cyclic. If $|H| = 2$ then H is generated by a reflection $\sigma' = \rho^i\sigma$: as $\rho^{-1}\sigma'\rho = \rho^{i-2}\sigma \neq e$ or σ', this is not normal in D_{2p}. If $|H| = p$ then $H = \langle \rho \rangle$, which is normal by Q6.

Therefore the normal subgroups of D_{2p} are $\{e\}, D_{2p}$ and $\langle \rho \rangle$.

(b) By lectures, the groups H for which there is a homomorphism from D_{2p} onto H are the groups D_{2p}/N, where $N \triangleleft D_{2p}$. Hence the groups H are C_1, D_{2p} and C_2.

8. (i) Yes, for example $\phi(x) = x^3$ (or $\phi(x) = x^9$).

(ii) No: the image of any homomorphism $C_{12} \to C_2 \times C_2$ must be cyclic (as it will be generated by the image of a generator of C_{12}), so it can't be surjective.

(iii) No. For suppose ϕ is a homomorphism from D_8 onto C_4. Then $\ker(\phi)$ has size 2. Let $K = \ker(\phi)$. As $K \triangleleft D_8$, K is not generated by a reflection (by Q6c), hence $K = \langle \rho^2 \rangle$. The First Isomorphism Theorem applied to ϕ implies that $D_8/K \cong \text{Im } \phi = C_4$. But it is not hard to check (I may well have done an example in lectures quotienting out D_{12} by $\langle \rho^2 \rangle$) that $D_8/\langle \rho^2 \rangle \cong C_2 \times C_2$, because each of the 4 right cosets K, $K\rho$, $K\sigma$, $K\rho\sigma$ has order 2, and this is a contradiction.

(iv) Yes: let $N = \langle \rho^2 \rangle \triangleleft D_8$. As in the previous part, $D_8/N \cong C_2 \times C_2$. Hence the map $x \to Nx$ is a homomorphism from D_8 onto $C_2 \times C_2$.

9. (a) If N is a subgroup of abelian G and $g \in G$, then for $n \in N$ we have $g^{-1}ng = g^{-1}gn = n \in N$, so $g^{-1}Ng \subseteq N$ for all g, and hence N is normal.

For $x, y \in G$, $(Nx)(Ny) = Nxy = Nyx = (Ny)(Nx)$. Hence G/N is abelian.

(b) $G = S_3, N = A_3$. Then N is abelian, as is $G/N \cong C_2$.

(c) Let $G = D_8$, $N = \{e, \rho^2, \sigma, \rho^2\sigma\}$ and $M = \langle \sigma \rangle$. I claim that $N \triangleleft G$. Assuming this, part (a) shows that $M \triangleleft N$ as N is abelian. But M is not normal in G.

It remains to convince you that N is a normal subgroup of G. If $g \in N$ then $g^{-1}Ng = Ng = N$, as $g^{-1}N = N$. If however $g \notin N$ then (because the size of G

is twice that of N) we must have that Ng is the other coset of N in G, that is, the elements of G that aren't in N (because distinct cosets are disjoint). Similar remarks apply for gN, and hence $Ng = gN$ and multiplying by g^{-1} we deduce $g^{-1}Ng = N$. Hence N is indeed normal in G.