KMB, 3/10/13

M2P2 Algebra II **Solutions to Sheet 1**

1. FTFFTFFTFTTTT if I got them all right...

2. (a) is easily checked using the permutations representing the group elements given in lectures.

(b) If $H = \langle\rho\rangle$ then as $|D_8 : H| = 2$, the right coset $H\sigma = \{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\}$ is equal to $D_8 - H$ which is $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$.

(c) Using (a) we get $\sigma\rho^n = \rho^{-n}\sigma$ for all n. Hence $\sigma_i\sigma_j = (\sigma\rho^m)(\sigma\rho^n)$ (for some m, n) $= \sigma\sigma\rho^{-m}\rho^n = \rho^{n-m}$, which is a rotation.

(d) e has order 1; ρ and ρ^3 have order 4; and the other 5 elements ρ^2, σ_i $(i = 1, 2, 3, 4)$ have order 2.

(e) The seven cyclic subgroups are $\langle e\rangle$, $\langle\rho\rangle$, $\langle\rho^2\rangle$, $\langle\sigma_i\rangle$ $(i = 1, 2, 3, 4)$.

3. Argue as in Q2.

4. $G(\Pi)$ contains a translation τ moving each D one place to the right, and a reflection σ in the horizontal line bisecting all of the D's. Now let's fix one of the D and call it D_0. Choose any $g \in G$ and consider where it takes D_0. It takes it to a random D, which we could call $g(D_0)$. Now there's definitely a translation which takes $g(D_0)$ back to D_0; if this translation is τ^{-n} then $\tau^{-n}g$ must send D_0 to itself. But the only symmetries of the D_0 are e and σ, so either $\tau^{-n}g$ or $\sigma\tau^{-n}g$ must send every point of D_0 to itself, and hence by Proposition 1.3 it must be the identity. It now follows easily that $g = \tau^n$ or $\tau^n\sigma$.

Finally, check geometrically that $\tau\sigma = \sigma\tau$. From this it is easy to check that $G(\pi)$ is abelian.

5. (a) Group has size 4 (identity, a rotation, two reflections), abelian.

(b) This group is infinite and non-abelian. The elements are of two kinds: firstly rotations by some angle $\theta \in [0, 2\pi)$, and secondly reflections about a line through the origin (and any such line will do). To see the group isn't abelian, see what happens if you choose a rotation and a reflection and then see if they commute in general.

(c) Same as (a).

(d) Just the one reflection, so cyclic of order 2 and abelian.

(e) Group is infinite (it contains all translations of the form $x \mapsto x + (m, n)$ with $m, n \in \mathbb{Z}$) and non-abelian (consider translation by the vector $(0, 1)$ and rotation by 90 degrees and check they don't commute).