Recall from last time:

Experimentation indicated that it was impossible to do an odd number of 2-cycles in S_3 & get back to the identity.

This is true: if $H \subseteq S_3$ is the subgroup generated by $(1\,2\,3)$ then $|H| = 3$ &

$$S_3 = H \cup H\tau, \quad \tau = (12)$$

Check $H\tau = \{(12), (13), (23)\}$

Check that if $g \in H$ & σ is any 2-cycle then $g\sigma \in H\tau$

Check that if $g \in H\tau$ & σ is a 2-cycle then $g\sigma \in H$.

Idea: ~~doing~~ "x transposition" switches between H & $H\tau$.

In general (n things)
is there a subgroup $H \subseteq S_n$ s.t.
① $|H| = \frac{1}{2}|S_n|$
② H contains no transpositions?

If so this will resolve the issues.

Now say $n \in \mathbb{Z}_{\geq 1}$
& say $\sigma \in S_n$.

Goal today: attach a "sign"

$$\text{sgn}(\sigma) \in \{+1, -1\}$$

to σ. $\text{sgn}(\sigma)$ is called the <u>signature</u> of σ. Here's the definition.

Do case $n=3$ first.

Let $\Delta = (x_1-x_2)(x_1-x_3)(x_2-x_3)$

where the x_i are "independent polynomial variables".

If $\sigma \in S_3$ then think of σ as a permutation of x_1, x_2, x_3

eg (12) sends $x_1 \to x_2$
$x_2 \to x_1$
$x_3 \to x_3$

& (123) sends $x_1 \to x_2$
$x_2 \to x_3$
$x_3 \to x_1$

If $\sigma \in S_3$, what is $\sigma(\Delta)$?

eg $\sigma = (12)$, $\sigma(\Delta) = (x_2-x_1)(x_2-x_3)(x_1-x_3)$
$= -\Delta$

$\sigma = (123)$, $\sigma(\Delta) = (x_2-x_3)(x_2-x_1)(x_3-x_1)$
$= \Delta = +\Delta$.

Each permutation in S_3 sends Δ to $\pm\Delta$.

$$\Delta = \prod_{1 \le i < j \le 3} (x_i - x_j)$$

& sign is counting number of pairs (i,j) with $i<j$ & $\sigma(i) > \sigma(j)$

sign is $+1$ if this number is even
-1 ——— odd.

$\mathrm{sgn}(\sigma)$ is this sign.

General case: $\Delta = \prod_{1 \le i < j \le n} (x_i - x_j)$

$= (x_1-x_2) \dots (x_1-x_n)(x_2-x_3) \dots (x_2-x_n) \dots (x_{n-1}-x_n)$

If $\sigma \in S_n$, then define $\mathrm{sgn}(\sigma)$ by

$$\sigma(\Delta) = \mathrm{sgn}(\sigma)\Delta$$
$$= \pm\Delta.$$

Upshot: have a function

$$\mathrm{sgn}: S_n \longrightarrow \{\pm 1\} = C_2$$

More notation:

Say $\sigma \in S_n$ is an <u>even</u> permutation if $\mathrm{sgn}(\sigma) = +1$
& say σ is <u>odd</u> if $\mathrm{sgn}(\sigma) = -1$.

$n=3$: we checked (12) was odd
& (123) was even.

Easy exercise: e, (123) & (132) are even, & (12), (13), (23) are all odd.

Key result:

<u>Prop^n 3.1</u> (a) If $x, y \in S_n$

then $\mathrm{sgn}(xy) = \mathrm{sgn}(x)\,\mathrm{sgn}(y)$.

(b) $\mathrm{sgn}(e) = +1$ & $\mathrm{sgn}(x^{-1}) = \mathrm{sgn}(x)$ $(= \mathrm{sgn}(x)^{-1})$

(c) If $\tau = (i\ j)$ is a 2-cycle then $\mathrm{sgn}(\tau) = -1$.

<u>Pf</u> (a) By def^n, $\mathrm{sgn}(x)\Delta = x(\Delta)$
& $\mathrm{sgn}(y)\Delta = y(\Delta)$.

Now $xy(\Delta) = x(y(\Delta)) = x(\mathrm{sgn}(y)\Delta)$

$$= \mathrm{sgn}(y).\,x(\Delta) = \mathrm{sgn}(y)\,\mathrm{sgn}(x)\,\Delta.$$

But $xy(\Delta) = \mathrm{sgn}(xy)\Delta$

$\therefore \mathrm{sgn}(xy) = \mathrm{sgn}(y)\,\mathrm{sgn}(x) = \mathrm{sgn}(x)\,\mathrm{sgn}(y)$.

(b) $e(\Delta) = \Delta \therefore \text{sgn}(e) = +1$.

If $x \in S_n$ then $x . x^{-1} = e$
& (a) $\Rightarrow \text{sgn}(x) . \text{sgn}(x^{-1}) = \text{sgn}(e) = +1$

$\therefore \text{sgn}(x) = \text{sgn}(x^{-1})^{-1} = \text{sgn}(x^{-1})$,

(c) Say $\tau = (i\ j)$. We need to work out $\tau(\Delta)$.

Recall ~~$\Delta = (x_1 - x_n)(x_2 - x_n) \dots$~~

$$\begin{aligned} \Delta = (x_1-x_2)(x_1-x_3) & \dots (x_1-x_n) \\ (x_2-x_3) & \dots (x_2-x_n) \\ & \vdots \\ & (x_{n-1}-x_n) \end{aligned}$$

Big question: when we apply τ, how many of those brackets will give us a sign change?

Let's assume $i<j$ (WLOG).

The brackets that give us a sign change are:

$(x_i - x_{i+1}), (x_i - x_{i+2}), \dots, (x_i - x_{j-1})$,

& $(x_{i+1} - x_j), (x_{i+2} - x_j), \dots, (x_{j-1} - x_j)$,

& finally $x_i - x_j$.

[Check: that's a complete list of the factors giving us sign changes]

Number of factors is: $j-i-1$ (1st line)
$+ \ j-i-1$ (2nd line)
$+ \ 1$

$= 2(j-i-1)+1 =$ odd number

$\therefore \text{sgn}(\tau) = (-1)^{\text{odd no}} = -1$ $\square$

Remark: other factors may move (eg $x_1 - x_i \to x_1 - x_j$) but don't give us a sign change.

We now know what signature of a 2-cycle is.

If we knew signature of a general d-cycle, we could work out signature of any permutation

eg in S_{10},

$$\operatorname{sgn}\big((1\ 3\ 6)(2\ 5)(7\ 9\ 8\ 10)\big)$$

$$= \operatorname{sgn}(1\ 3\ 6) \times \operatorname{sgn}(2\ 5) \times \operatorname{sgn}(7\ 9\ 8\ 10)$$

Propn 3.2 If $c = (a_1\ a_2 \cdots a_d)$ is a d-cycle in S_n

then $c = (a_1\ a_d)(a_1\ a_{d-1}) \ldots (a_1\ a_3)(a_1\ a_2)$.

Pf: The RHS product sends a_1 to a_2
a_2 to a_3
a_3 to a_4
$\vdots$
a_{d-1} to a_d
a_d to a_1

& fixes everything else

$\therefore$ RHS $= c$. $\square$

Cor 3.3 $\operatorname{sgn}(d\text{-cycle}) = (-1)^{d-1}$

Pf If c is a d-cycle
can write $c = \tau_1 \tau_2 \ldots \tau_{d-1}$
τ_i all 2-cycles, by Propn 3.2.

$\therefore \operatorname{sgn}(c) = \operatorname{sgn}(\tau_1)\operatorname{sgn}(\tau_2) \ldots \operatorname{sgn}(\tau_{d-1})$
by repeated applications of 3.1(a)

& $\operatorname{sgn}(\tau_i) = -1$ by 3.1(c)

$\therefore \operatorname{sgn}(c) = (-1)^{d-1}$. $\square$

So (back to S_{10} example)

$$\operatorname{sgn}\big((1\ 3\ 6)(2\ 5)(7\ 9\ 8\ 10)\big)$$

$$= (-1)^2 \times (-1)^1 \times (-1)^3 = (-1)^6 = +1.$$