Proof of JCF theorem is in 3 stages

1. ~~Exp~~ Can add vector spaces together $V \oplus W$ (like adding matrices together) (like products of groups) & can break big vector space V up as $V = V_1 \oplus V_2 \oplus \dots \oplus V_s$ (general theory)

2. $T: V \to V$ linear map. We'll break V up as in 1
$$V = V_1 \oplus V_2 \oplus \dots \oplus V_s$$
s.t. $T(V_i) \subseteq V_i$ & if $T_i \; V_i \to V_i$ is induced restriction map then T_i has only 1 eigenvalue

3. Prove JCF thm for linear maps with only 1 eigenvalue.

Today let's start on 3.

Suppose A is an $n \times n$ matrix, & suppose its char poly is $(x-\lambda)^n$

If A has a JCF, it had better be

$$J = \underbrace{J_1(\lambda) \oplus J_1(\lambda) \oplus \dots \oplus J_1(\lambda)}_{a_1} \oplus J_2(\lambda)$$

$$\oplus J_2(\lambda) \oplus \dots \oplus J_2(\lambda) \oplus J_3(\lambda) \oplus \dots$$

a_2 copies of $J_2(\lambda)$

Notation:

$$J = J_1(\lambda)^{a_1} \oplus J_2(\lambda)^{a_2} \oplus \dots \oplus J_r(\lambda)^{a_r}.$$

Recall $J_1(\lambda) = (\lambda)$, $J_2(\lambda) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ etc

$$\therefore J = \begin{pmatrix} \underbrace{\lambda \quad \lambda \quad \ddots \quad \lambda}_{a_1} & & & \\ & \underbrace{\begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix} \; 0 \; \begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix} \; \ddots \; \begin{matrix} \lambda & 1 \\ 0 & \lambda \end{matrix}}_{\text{total no of rows is } 2a_2} & & \\ & & \begin{matrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{matrix} & \\ & & & \ddots \end{pmatrix}$$

Note that $J - \lambda I$

$= J_1(0)^{a_1} \oplus J_2(0)^{a_2} \oplus \cdots \oplus J_r(0)^{a_r}$.

In particular, we know

$(J - \lambda I)^r = \underset{\sim}{0} \oplus \underset{\sim}{0} \oplus \cdots \oplus \underset{\sim}{0}$

$= \underset{\approx}{0}$ = zero matrix

Let's consider $(J - \lambda I)^{r-1}$.

We can work this out using 11.2 (4)

11.2 (4) said this.

If $X = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

then $X^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$X^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

& $X^4 = \underset{\approx}{0}$

Prop^n 11.6 Let A be an $n \times n$ matrix with char poly $(x-\lambda)^n$.

If A has a JCF, then its unique (up to re-ordering the factors) & furthermore, JCF can easily be determined by

$\operatorname{rank}((A-\lambda I)^i)$ for $1 \le i \le n$.

Pf $A \sim J = J_1(\lambda)^{a_1} \oplus \cdots \oplus J_r(\lambda)^{a_r}$

by assumption. Need to check that I can read off all the a_i from A alone.

Know $(J-\lambda I)^r = \underset{\sim}{0}$ $\therefore$ $(A-\lambda I)^r = \underset{\sim}{0}$

What is $(J-\lambda I)^{r-1}$?

All the $J_d(\lambda)$, $d \le r-1$ will become 0.

$\therefore (J-\lambda I)^{r-1}$

$$= \begin{pmatrix} O & & \\ & (J_r(\lambda)-\lambda I)^{r-1} & \\ & & (J_r(\lambda)-\lambda I)^{r-1} \end{pmatrix}$$

$$(J_r(\lambda)-\lambda I)^{r-1} \rightarrow \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & & 0 \\ 0 & \cdots & & & 0 \end{pmatrix}$$

$$(J_r(\lambda)-\lambda I)^{r-1} \rightarrow \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & & 0 \\ 0 & \cdot & & 0 \end{pmatrix}$$

$\therefore$ rank of $(J-\lambda I)^{r-1}$ = number of $J_r(\lambda)$ blocks

$= a_r$

$\therefore$ rank of $(A-\lambda I)^{r-1} = a_r$ ($(J-\lambda I)^{r-1}$ is similar to $(A-\lambda I)^{r-1}$)

What is $(J-\lambda I)^{r-2}$?

All the $J_d(\lambda)$, $d \le r-2$, will become O

$J_{r-1}(\lambda)$ will give a rank of 1

& $J_r(\lambda)$ will give us a rank of 2

$$(J_r(\lambda)-\lambda I)^{r-2}$$

$$= \begin{pmatrix} & 0 & 1 & 0 \\ & & 0 & 1 \\ O & & & 0 \end{pmatrix} \leftarrow \text{rank} = 2.$$

$\therefore (J-\lambda I)^{r-2}$ has rank $a_{r-1}+2a_r$

$\therefore$ rank of $(A-\lambda I)^{r-2}$ is $a_{r-1}+2a_r$

& We worked out a_r from $rk(A-\lambda I)^{r-1}$

$\therefore$ can now work out a_{r-1}.

Continue in this way!

Conclude that we can compute all the a_i from rank of $(A-\lambda I)^j$, $1\le j\le r\le n$.

$\square$

Upshot: so far we know that if A has 1 eigenvalue, it has at most one JCF.

Now let's prove it has at least one!

Thm 11.7

Say V is an n-dimensional vector space & say $S: V\to V$ is a linear map st char poly of S is x^n.

Then V has a basis $\mathcal{B}$ s.t.

$$[S]_{\mathcal{B}} = J_{n_1}(0)\oplus J_{n_2}(0)\oplus\ldots\oplus J_{n_s}(0).$$

(ie JCF exists if only 1 eval & it's 0)

__Proof__ Need a basis B st

$$[S]_B = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & 0 & \\ & & 0 & 0 & & \\ & & & 0 & 1 & \\ & 0 & & & 0 & \ddots \\ & & & & & 0 \end{pmatrix}$$

diagonal $= 0$'s & 1's.

Let, call the basis ~~the~~

$B = \{ v_{n}, v_{n-1}, \dots, v_2, v_1 \}$

(order it backwards)

Then $Sv_1 = v_2$, $Sv_2 = v_3 \dots Sv_{n_1} = 0$ etc.

Want a basis that behaves like this

More formally, what we must prove is that we can find a basis of following form:

$$\{ w_1, Sw_1, S^2 w_1, \dots, S^{n_1 - 1} w_1, w_2, Sw_2, \dots, S^{n_2 - 1} w_2, w_3, \dots, S^{n_s - 1} w_s \} \quad (**)$$

where $S^{n_i} w_i = 0$ for $1 \leq i \leq s$.

If we have this, then write basis backwards & we've proved the result.

We will prove that such a basis exists, by induction.

Note first that char poly of S is x^n $\therefore$ by Cayley-Hamilton, $S^n = \underset{\approx}{0}$.

Will do this next time, because it is genuine pain.