Chapter 6 – the story so far

* – saw homomorphisms

$$\varphi : G \longrightarrow H$$
$$\varphi(xy) = \varphi(x)\varphi(y)$$

* saw <u>normal</u> subgroups

$N \lhd G$ means
N is a subgroup & $g^{-1}Ng = N$ for all $g \in G$.

* Seen subgroups are normal & some aren't.

eg $\langle (1\,2) \rangle \leq S_3$ was <u>not</u> normal.

<u>Propn 6.5</u> Let $\varphi : G \to H$ be a group hom. Then
$$\ker(\varphi) \lhd G.$$

<u>Proof</u> Set $N = \ker(\varphi) = \{ g \in G : \varphi(g) = e_H \}$.

Need to check $g^{-1}Ng = N$ for all $g \in G$.

By 6.4, only need to check $g^{-1}Ng \subseteq N$.

So say $x \in g^{-1}Ng$ is arbitrary.

Need to check $\varphi(x) = e_H$. If that's true, then $x \in N$ & we're done.

Know $x = g^{-1}ng$ for some $n \in N$ (def^n of x)

$\therefore \varphi(x) = \varphi(g^{-1}ng)$ (clear)

$= \varphi(g^{-1})\varphi(n)\varphi(g)$ (φ is a gp hom)

$= \varphi(g^{-1})e_H \cdot \varphi(g)$ ($n \in \ker(\varphi)$)

$= \varphi(g^{-1})\varphi(g)$

$= \varphi(g^{-1}g)$ (φ is a gp hom)

$= \varphi(e_G)$

$= e_H$ (by 6.1)

$\therefore x \in N$ & we're done $\square$

Consequence: new source of normal subgps: namely the kernels of gp homs.

Back to examples of gp homs.

* Identity map $i: G \to G$
  $i(x) = x$.

$\ker(i) = \{e_G\}$

$\therefore \{e_G\} \lhd G$.

* Trivial map $G \to H$
  $\varphi(x) = e_H \quad \forall x \in G$

$\ker(\varphi) = G \quad \therefore G \lhd G$.

* $\mathrm{sgn}: S_n \to C_2$

$\ker(\mathrm{sgn}) = \{\sigma \in S_n : \mathrm{sgn}(\sigma) = +1\}$

$= A_n$

$\therefore A_n \lhd S_n$.

* $\varphi: D_{2n} \to \{\pm 1\}$

$\varphi(\rho^i \sigma^j) = (-1)^j$

$\ker(\varphi) =$ rotations $\quad ((-1)^j = 1 \Rightarrow j$ even $\Rightarrow \sigma^j = e)$

$= \langle \rho \rangle$

$\therefore \langle \rho \rangle \lhd D_{2n}$.

[* Recall $D_6 \cong S_3$
$\rho \leftrightarrow (123)$
$\sigma \leftrightarrow (12)$

$\therefore \langle (123) \rangle \lhd S_3$. [mull this over!]]

Factor Groups, a.k.a. quotient groups

Plan: "try to divide a group by a subgroup & get a quotient group"

· More rigorous plan:

If G is a group & H is a subgroup, we can consider the set of right cosets for H in G.

Call that set G/H.

An element of G/H is a set Hg - a subset of G

$$G/H = \{ H, Hx_2, Hx_3, \ldots \}$$

Note: if G & H are finite groups, then $|G/H| = |G|/|H|$; so maybe notation isn't too bad.

Big Q: G & H are groups - can we make G/H into a group?

Unfortunately we cannot in general do this in a natural way.

Let's try, & let's fail.

Try: ~~must~~ need a multiplication on G/H.

GUESS: define $(Hx).(Hy) = Hxy$.

However, this def^n doesn't make sense.

The problem is this: you can have $x_1 \neq x_2$ in G, but with $Hx_1 = Hx_2$.

So here is an issue.

Say $Hx_1 = Hx_2$

Say $Hy_1 = Hy_2$.

Does $Hx_1y_1 = Hx_2y_2$?

If that's true, our defⁿ is fine.

<u>Example</u>

$G = S_3$, $H = \langle (12) \rangle = \{e, (12)\}$

Set $x_1 = (1\,2\,3)$.

Then $Hx_1 = \{ hx_1 : h \in H \}$

$= \{ (1\,2\,3), (12)(123) \}$

$= \{ (1\,2\,3), (2\,3) \} \subseteq G$.

So (recall distinct cosets are disjoint)

if $x_2 = (23)$

then $x_2 \in Hx_1$

$\therefore Hx_1 = Hx_2$

Set $y_1 = x_1$, $y_2 = x_2$

$Hy_1 = Hy_2$

But $Hx_1y_1 = H(x_1)^2 = H(132) \neq H$

& $Hx_2y_2 = H(x_2)^2 = He = H$

<u>FIX</u>

Assume H is <u>normal</u>. Then it will all work.

Lemma 6.6

Say G is a group

& $N \trianglelefteq G$ is a normal subgroup of G.

Then if $x_1, y_1, x_2, y_2 \in G$

& if $Nx_1 = Nx_2$ & $Ny_1 = Ny_2$

then $Nx_1y_1 = Nx_2y_2$

Proof Recall that $N \trianglelefteq G$ means

$$g^{-1}Ng = N \text{ for all } g \in G.$$

$\times$ on left by g: get $\underline{\underline{Ng = gN}}$ if N is normal.

So
$$\begin{aligned}
Nx_2y_2 &= (Nx_2)y_2 \\
&= (x_2N)y_2 && (N \text{ normal}) \\
&= x_2(Ny_2) && (\text{associativity}) \\
&= x_2(Ny_1) && (\text{assumption}) \\
&= (x_2N)y_1 && (\text{associativity}) \\
&= (Nx_2)y_1 && (N \text{ normal}) \\
&= (Nx_1)y_1 && (\text{def}^{n}) \\
&= Nx_1y_1 \quad \square
\end{aligned}$$

Coming up: if $N \trianglelefteq G$ then G/N is a group.