KMB, 28th November 2013

M2PM2 Algebra II, Problem Sheet 9

1. For each of the following matrices A, find an invertible matrix P over $\mathbb{C}$ such that $P^{-1}AP$ is upper triangular:

$$A = \begin{pmatrix} 4 & 1 \\ -1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}$$

2. Let A be an $n \times n$ matrix, and suppose that the only eigenvalue of A in $\mathbb{C}$ is 0. Prove that $A^n = 0$.

3. Let $a_0, a_1, \ldots, a_{n-1} \in \mathbb{R}$, and let A be the $n \times n$ matrix

$$A = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ & & & \cdots & & \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{pmatrix}$$

Prove that the characteristic polynomial of A is $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. (*Hint:* Try induction.)

4. In this question you can use Q3 and Cayley–Hamilton.
(a) Find a 3×3 matrix which has characteristic polynomial $x^3 - 7x^2 + 2x - 3$.
(b) Find a 3×3 matrix A such that $A^3 - 2A^2 = I$.
(c) Find an invertible 4×4 matrix B such that $B^{-1} = B^3 + I$.
(d) Find a real 4×4 matrix C such that $C^2 + C + I = 0$.
(e) For each $n \geq 2$ find an $n \times n$ matrix D with real coefficients such that $D^n = I$ but $D \neq I$.

5. Let A be an arbitrary $n \times n$ matrix. Which of the following quantities are invariants of A (i.e. are the same for any matrix which is similar to A) ? Give brief justifications for your answers.

(i) $\text{rank}(A^3 - I)$
(ii) $\text{trace}(A + A^5)$
(iii) $c_1(A)$, the sum of the entries in the first column of A
(iv) $\text{rank}(A - A^T)$
(v) $\text{trace}(2A - A^T)$.

6. Suppose that λ is an eigenvalue of a block-diagonal matrix $A = A_1 \oplus \cdots \oplus A_k$. Prove that the geometric multiplicity of λ for A is equal to the sum of its geometric multiplicities for each A_i. (In other words prove that $\dim E_\lambda(A) = \sum_1^k \dim E_\lambda(A_i)$, where $E_\lambda(A)$ and $E_\lambda(A_i)$ are the λ-eigenspaces of A and A_i.)

7. (i) Write down all the possible Jordan Canonical Forms having characteristic polynomial $x(x + 1 + i)^2(x - 3)^3$.

(ii) Calculate the number of non-similar Jordan Canonical Forms having characteristic polynomial $x^3(x - 1)^6$.

8. Find the JCFs of the following matrices:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ -1 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -1 & 0 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & i & 2 \\ 0 & 0 & 0 & 0 & 0 & i \end{pmatrix}$$

9. Let $J_n(\lambda)$ be a Jordan block. Prove that the matrix $J = J_n(\lambda) - \lambda I$ is similar to its transpose. (*Hint (if needed):* Consider the linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ defined by $T(v) = Jv$, and try to find bases E, F such that $[T]_E = J$, $[T]_F = J^T$.)

Deduce that $J_n(\lambda)$ is similar to its transpose.

10. Using Q9 and the JCF theorem, prove that every square matrix over $\mathbb{C}$ is similar to its transpose.

11. If A is an $n \times n$ matrix, a square root of A is defined to be an $n \times n$ matrix B such that $B^2 = A$.

(i) Give an example of a matrix that has no square root.

(ii) Using the JCF theorem, or otherwise, prove that every invertible matrix A over $\mathbb{C}$ has a square root.

12.‡ Say U, V, W, X, Y and Z are vector spaces and we have the following linear maps between them:

$$\begin{array}{ccccc} U & \xrightarrow{\alpha_1} & V & \xrightarrow{\beta_1} & W \\ \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\epsilon} \\ X & \xrightarrow{\alpha_2} & Y & \xrightarrow{\beta_2} & Z \end{array}$$

Suppose that:

- α_i is injective and β_i is surjective ($1 \le i \le 2$)
- the image of α_i equals the kernel of β_i ($1 \le i \le 2$)
- $\delta \circ \alpha_1 = \alpha_2 \circ \gamma$ and $\epsilon \circ \beta_1 = \beta_2 \circ \delta$ (we say "both squares commute").

Prove the following:

(i) If γ and ϵ are injective, then so is δ.

(ii) If γ and ϵ are surjective, then so is δ.

(iii) If δ is an isomorphism, then γ is injective, ϵ is surjective, and furthermore ϵ is injective iff γ is surjective!