M2PM2 Algebra II, Problem Sheet 8

1. Define a relation on $n \times n$ matrices by $A \sim B$ if and only if A and B are similar. Prove that $\sim$ is an equivalence relation.

2. Let E, F be the following bases of $\mathbb{R}^2$: $E = \{(1, 0), (0, 1)\}$, $F = \{(1, 3), (2, 5)\}$.
   (i) Find the change of basis matrix P from E to F.
   (ii) Find the change of basis matrix Q from F to E.
   (iii) Check that $Q = P^{-1}$.
   (iv) Let $v = (a, b) \in \mathbb{R}^2$. Write down the column vectors $[v]_E$ and $[v]_F$, and check that $[v]_E = P[v]_F$.
   (v) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(x_1, x_2) = (2x_2, 3x_1 - x_2)$. Check that $[T]_F = P^{-1}[T]_E P$.

3. Find the determinants of the following linear transformations T:
   (i) $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, \; -x_1 - 3x_3, \; x_2 + x_3).$$

   (ii) Let V be the vector space of polynomials of degree at most 3 over $\mathbb{R}$, and define $T : V \to V$ by $T(p(x)) = p(1 + x) - p'(1 - x)$ for all $p(x) \in V$.

   (iii) Let V be the vector space of all 2×2 matrices over $\mathbb{R}$, let $M = \begin{pmatrix} 1 & -2 \\ 1 & 4 \end{pmatrix}$, and define $T : V \to V$ by $T(A) = MA$ for all $A \in V$.

4. For T as in each of parts (ii) and (iii) of Q3, find the eigenvalues of T, and find a basis for each eigenspace. Is there a basis B of V such that the matrix $[T]_B$ is diagonal ?

5. (a) For each of the following linear transformations $T : V \to V$, find its eigenvalues, find their algebraic and geometric multiplicities, and determine whether V has a basis consisting of eigenvectors of T:
   (i) $V = \mathbb{R}^3$, $T(x_1, x_2, x_3) = (-x_1 + x_2 - x_3, \; -4x_2 + 6x_3, \; -3x_2 + 5x_3)$.
   (ii) V the vector space of polynomials over $\mathbb{R}$ of degree at most 2, and $T(p(x)) = x(2p(x + 1) - p(x) - p(x - 1))$ for all $p(x) \in V$.

(b) For which values of a, b, c is the matrix

$$A = \begin{pmatrix} -1 & a & b \\ 0 & 1 & c \\ 0 & 0 & -1 \end{pmatrix}$$

diagonalisable (i.e. $\exists \, P$ such that $P^{-1}AP$ is diagonal) ?

6. Let V be an n-dimensional vector space over $\mathbb{R}$, $n \geq 1$. Define $GL(V)$ to be the set of all invertible linear transformations from $V \to V$.
   (i) Prove that $GL(V)$ is a group under composition.
   (ii) Prove that the function $\det : GL(V) \to (\mathbb{R}^*, \times)$ is a surjective homomorphism.
   (iii) Prove that $GL(V) \cong GL(n, \mathbb{R})$. (Recall that $GL(n, \mathbb{R})$ is the group of all invertible $n \times n$ matrices over $\mathbb{R}$, under matrix multiplication.)