KMB, 5th November 2013

M2PM2 Algebra II Problem Sheet 6

1. (a) Let G be a cyclic group, and let N be a subgroup of G. Prove that $N \triangleleft G$ and the factor group G/N is also cyclic.

(b) Let $n \in \mathbb{N}$, and let $n\mathbb{Z} = \langle n \rangle$ be the cyclic subgroup of $(\mathbb{Z}, +)$ generated by n. Use the First Isomorphism Theorem to prove that the factor group $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $(\mathbb{Z}_n, +)$, the group of residue classes modulo n under addition.

(c) Say G and H are groups, and $\phi : G \times H \to H$ is defined by $\phi(g, h) = h$. Prove that ϕ is a homomorphism, and prove that the kernel of ϕ is isomorphic to G.

(d) Find a non-abelian group G, and a normal subgroup N of G, such that $G/N \cong C_2 \times C_2 \times C_2$.

2. (i) Prove that if there is a surjective homomorphism from S_n onto C_r, then r must be 1 or 2.

(ii) (quite tricky!) Prove that if N is a normal subgroup of S_n such that the factor group S_n/N is cyclic, then $N = S_n$ or A_n.

3. For each of the following groups G, find all groups H (up to isomorphism) such that there is a surjective homomorphism from G to H:

$$G = D_8, \quad G = D_{12}, \quad G = S_4.$$

Slight warning: S_4 is tricky – but satisfying!

4. In lectures we defined dihedral groups by looking at symmetry groups of certain 2-dimensional shapes. These tricks are not confined to two dimensions however. In this question we investigate the symmetries of a regular tetrahedron. The general story in 3 dimensions is the same as the 2-dimensional case: an isometry is a map $f : \mathbb{R}^3 \to \mathbb{R}^3$ such that $d(x, y) = d(f(x), f(y))$, where d is the usual distance function on $\mathbb{R}^3$: $d(x, y)$ is the distance from x to y.

Examples of isometries $\mathbb{R}^3 \to \mathbb{R}^3$: rotation about a line, reflection in a plane.

Now let Π denote a regular tetrahedron, and let's label the vertices 1, 2, 3 and 4.

(a) Prove that there are at least 12 rotations mapping Π to Π. Hint: check that as well as the identity, there are 8 rotations of order 3 that fix a vertex, and 3 rotations of order 2 that fix the midpoints of two opposite edges.

(b) Prove that there are exactly 12 rotations mapping Π to Π. Hint: there are four faces, and each face can be orientated in three ways. Look at how we did this sort of thing in the 2-d case.

(c) Prove that the group of rotations that preserve Π is isomorphic to A_4. Hint: consider permutations of the vertices.

(d) Prove that the full symmetry group of the tetrahedron is isomorphic to S_4. Hint: check that there is a reflection that switches two vertices and fixes the other two.

5. Let R be the set of rotations in the symmetry group of a cube.

(a) By writing down as many rotation symmetries as you can think of, show that $|R| \geq 24$.

(b) By counting the possibilities for the 'bottom' face and the 'front' face, show that $|R| \leq 24$.

(c) By considering each rotation as a permutation of the 4 long diagonals of the cube, show that $R \cong S_4$.

6.‡ What do you think the group of rotations of an octahedron is? A dodecahedron? An icosahedron? A 10-dimensional hypercube? A hecatonicosachoron?