

M2PM2 Algebra II, Problem Sheet 5

1. Which of the following functions ϕ are group homomorphisms? For those which are group homomorphisms, find $\text{Im } \phi$ and $\text{Ker } \phi$.

- $\phi : C_{12} \rightarrow C_{12}$ defined by $\phi(x) = x^3 \quad \forall x \in C_{12}$
- $\phi : S_4 \rightarrow S_4$ defined by $\phi(x) = x^3 \quad \forall x \in S_4$
- $\phi : (\mathbb{Z}, +) \rightarrow (\mathbb{Z}_n, +)$ defined by $\phi(x) = [x]_n \quad \forall x \in \mathbb{Z}$
- $\phi : (\mathbb{R}_{>0}, \times) \rightarrow (\mathbb{R}_{>0}, \times)$ defined by $\phi(x) = \sqrt{x} \quad \forall x \in \mathbb{R}_{>0}$
- $\phi : (\mathbb{Z}_6, +) \rightarrow (\mathbb{Z}_7, +)$ defined by $\phi([x]_6) = [x]_7 \quad \forall [x]_6 \in \mathbb{Z}_6$
- $\phi : (\mathbb{Z}_6, +) \rightarrow (\mathbb{Z}_7^*, \times)$ defined by $\phi([x]_6) = [2^x]_7 \quad \forall [x]_6 \in \mathbb{Z}_6$

Recall the notation: $[x]_n$ stands for the residue class of x modulo n , and $\mathbb{R}_{>0}$ stands for the set of positive real numbers.

2. Let G be a group.

(a) We say that elements $x, y \in G$ are *conjugate* (or more precisely are *conjugate in G*) if there exists $g \in G$ with $g^{-1}xg = y$. Prove that conjugacy is an equivalence relation.

(b) The equivalence classes in (a) are called *conjugacy classes*. Prove that a subgroup H of G is normal iff it is a union of conjugacy classes.

3. (a) Let σ be the 5-cycle $(1\ 2\ 3\ 4\ 5) \in S_5$. Prove that σ is conjugate to σ^2 in S_5 .

(b) Now note that $\text{sgn}(\sigma) = +1$ so $\sigma \in A_5$. Is σ conjugate to σ^2 in A_5 ? [hint: if $x\sigma x^{-1} = \sigma^2$ then there are only five possibilities for $x(1)$, and $x(1)$ determines x because $x\sigma x^{-1} = \sigma^2$].

4. Let G be a group, and suppose M and N are normal subgroups of G . Show that $M \cap N$ is a normal subgroup of G .

5. Let G be a group, let $g \in G$ be an element of this group, and define a homomorphism $\phi : G \rightarrow G$ by $\phi(x) = g^{-1}xg$. Prove that ϕ is an isomorphism. Give an example to show that ϕ may not be the identity map.

6. Let $G = D_{2n} = \{e, \rho, \dots, \rho^{n-1}, \sigma, \rho\sigma, \dots, \rho^{n-1}\sigma\}$, where $n \geq 3$ and ρ, σ satisfy the usual equations $\rho^n = \sigma^2 = e$, $\sigma\rho = \rho^{-1}\sigma$. Recall from the test that $\sigma\rho^k = \rho^{-k}\sigma$ for all integers k .

(a) Let r be a fixed integer. Prove that the cyclic subgroup $\langle \rho^r \rangle$ is a normal subgroup of D_{2n} .

(b) Let r be a fixed integer. Prove that $\langle \rho^r \sigma \rangle$ is not a normal subgroup of D_{2n} .

7. Let p be a prime number greater than 2.

(a) Prove that the dihedral group D_{2p} has exactly three different normal subgroups.

(b) Find all groups H (up to isomorphism) such that there is a surjective homomorphism from D_{2p} onto H .

8. Does there exist a surjective homomorphism

- (i) from C_{12} onto C_4 ?
- (ii) from C_{12} onto $C_2 \times C_2$?
- (iii) from D_8 onto C_4 ?
- (iv) from D_8 onto $C_2 \times C_2$?

Give reasons for your answers.

9. (a) Show that if G is an abelian group, and N is a subgroup of G , then $N \triangleleft G$ and the factor group G/N is abelian.

(b) Give an example of a non-abelian group G with a normal subgroup N such that both N and G/N are abelian.

(c) Give an example of a group G with subgroups M, N such that $N \triangleleft G$ and $M \triangleleft N$, but M is not normal in G .

10.[‡] Say G and H are finite groups, and there exists a surjective group homomorphism from $G \times G$ to $H \times H$. Must there exist a surjective group homomorphism from G to H ?