KMB, 29th October 2013

M2PM2 Algebra II Problem Sheet 4

1. (a) Prove that S_5 does not have a subgroup which is isomorphic to D_{20}.

(b) Prove that S_5 *does* have a subgroup which is isomorphic to D_{12}.

The next three questions lead you through a proof of what is called the "Primary Decomposition Theorem" for finite abelian groups. The proof assumes *the Structure Theorem, namely that every finite abelian group is a product of cyclic groups. The questions are quite long, sometimes tough, and skippable if you want.*

2. Let p be a prime and let $r \in \mathbb{N}$.

(a) List the orders of elements of the cyclic group C_{p^r}, and find the number of elements of each order in this group.

(b) Let G be the direct product $C_{p^r} \times C_{p^r} \times \cdots \times C_{p^r}$, where there are k factors C_{p^r}. (So G is abelian of size p^{rk}.) Prove that the number of elements in G of order p^r is equal to

$$p^{k(r-1)}(p^k - 1).$$

(*Hint*: count the elements *not* of order p^r!)

3. Let p be a prime and let $r \in \mathbb{N}$. Let $\mathbf{a} = (a_1, \ldots, a_k)$ and $\mathbf{b} = (b_1, \ldots, b_l)$ be sequences of positive integers such that $a_1 \geq a_2 \geq \cdots \geq a_k$, $b_1 \geq b_2 \geq \cdots \geq b_l$, and $\sum_1^k a_i = \sum_1^l b_i = r$. Define abelian groups $G_\mathbf{a}$ and $G_\mathbf{b}$ of size p^r as follows:

$$G_\mathbf{a} = C_{p^{a_1}} \times \cdots \times C_{p^{a_k}}, \quad G_\mathbf{b} = C_{p^{b_1}} \times \cdots \times C_{p^{b_l}}.$$

Prove that $G_\mathbf{a} \cong G_\mathbf{b}$ if and only if $\mathbf{a} = \mathbf{b}$ (meaning that $k = l$ and $a_i = b_i$ for all i).

4. Deduce from Q3 (and the Structure Theorem for Finite Abelian Groups, which you can *assume* for this question) the famous Primary Decomposition Theorem for abelian groups: every finite abelian group is isomorphic to a *unique* (up to re-ordering) direct product of the form

$$C_{p_1^{r_1}} \times \cdots \times C_{p_k^{r_k}},$$

where p_i are primes, $p_1 \geq p_2 \geq \cdots \geq p_k$ and $r_i \in \mathbb{N}$.

Hint for the next two questions: mimic what I did for the case $|G| = 6$, and remember sheet 2 question 6b.

5. Let G be a non-abelian group such that $|G| = 10$. Prove that $G \cong D_{10}$.

6. Let G be a non-abelian group such that $|G| = 14$. Prove that $G \cong D_{14}$.

7. What do you think the general theorem is for the isomorphism classes of groups of order $2p$ with p an odd prime?

8. Let A, B be the following matrices over the complex numbers:

$$A = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(a) Show that $A^4 = B^4 = I$, $A^2 = B^2$ and $BA = A^3B$.

(b) Deduce that the set $\{A^rB^s \ : \ r, s \in \mathbb{Z}\}$ consists of exactly 8 matrices, and write them down.

(c) Let Q_8 be the set of matrices in (b). Prove that Q_8 is a subgroup of $GL(2, \mathbb{C})$.

(d) Prove that $Q_8 \not\cong D_8$.

The next two questions show how to complete the classification of groups of order 8. Recall that (assuming the structure theorem) we have figured out what all abelian groups of order 8 are, so the issue is what the non-abelian ones are.

9. Let G be a non-abelian group such that $|G| = 8$.

(a) Prove that G has an element x of order 4.

(b) Given that G has an element y such that y has order 2 and $y \neq x^2$, prove that $G \cong D_8$. (*Hint: try to copy the proof in lectures for groups of size 6.*)

10. Prove that up to isomorphism, the only non-abelian groups of size 8 are D_8 and Q_8.

11.[‡] Prove that the only normal subgroups of A_n, $n \geq 5$, are $\{e\}$ and A_n. We say A_n, $n \geq 5$, is a *simple* group.