KMB, 22/10/13

M2PM2 Algebra II **Problem Sheet 3**

1. Consider the 5-cycle $g = (12345)$ in S_5.

(i) Can g be written as a product of four 2-cycles?

(ii) Can g be written as a product of 2013 2-cycles?

(iii) Can g be written as a product of two 2-cycles?

2. For each pair of the following four groups, either prove that they are isomorphic or prove that they are not! $C_1 \times C_{24}$, $C_2 \times C_{12}$, $C_3 \times C_8$, $C_4 \times C_6$.

3.

(a) Find, with proof, an abelian group with at least 100 elements of order 3.

(b) Find, with proof, a non-abelian group with at least 100 elements of order 3.

4.

(a) Find the number of elements of order 3 in A_5.

(b) Find the number of elements of order 3 in A_6.

(c) What is the smallest n for which there is an element of order 8 in A_n?

Justify your answers.

5. (a) Write down two even permutations in S_9, both of which have order 12 and send $1 \to 6$, $2 \to 3$ and $9 \to 5$.

(b) Let $I = \{1, 2, \ldots, 2n\}$ and let $f : I \to I$ be the permutation sending i to $2n + 1 - i$ for all $i \in I$. Find the signature $\operatorname{sgn}(f)$ in terms of n.

6. (a) Let G_1, G_2 be groups. Prove that $G_1 \times G_2 \cong G_2 \times G_1$.

(b) Let G_1, G_2, H_1, H_2 be groups such that $G_1 \cong H_1$ and $G_2 \cong H_2$. Prove that $G_1 \times G_2 \cong H_1 \times H_2$.

7. Up to isomorphism, how many different abelian groups are there

(a) of size 30 ?

(b) of size 31 ?

(c) of size 32 ?

You may assume the Structure Theorem for finite abelian groups.

8. Use direct products (or any other method!) to give examples of groups G with the following properties:

(i) $|G| = 2^n$ and $x^2 = e$ for all $x \in G$, where n is an arbitrary positive integer.

(ii) $|G| > 8$, G is non-abelian, and $x^4 = e$ for all $x \in G$

(iii) G is infinite and non-abelian, and G has a subgroup H such that $|G : H| = 2$ and H is abelian (recall $|G : H|$ is the *index* of H in G, i.e. the number of distinct right cosets of H in G).

9.‡ Let's say a group G is *indecomposable* if $|G| > 1$ and G is not isomorphic to $H \times K$ for any groups H and K of size greater than 1. Can there be indecomposable abelian groups G_1, G_2, G_3, G_4 and G_5 with $G_1 \times G_2$ isomorphic to $G_3 \times G_4 \times G_5$?