KMB, 3/10/13

M2PM2 Algebra II Problem Sheet 1

Questions marked with ‡ are challenge problems. They are probably harder than most exam questions and can be safely ignored unless you're looking for a challenge. There will be one per sheet until I run out of ideas. No answers will be supplied for the ‡ questions.

1. (*Revision!*) Decide whether each of the following statements is true or false. Throughout, G is a group.

1. If we can find elements g, h in G such that $gh = hg$ then G is abelian.
2. If G is cyclic then G is abelian.
3. If G is not cyclic then G is not abelian.
4. If G is infinite then no element of G has finite order.
5. If $G = S_n$ then the size of every subgroup of G divides $n!$.
6. If $G = S_n$ then no element of G has order greater than n.
7. If the order of every non-identity element of G is a prime number then G is cyclic.
8. If $G = \langle g \rangle$ is an infinite cyclic group, then g and g^{-1} are the only generators of G.
9. If G is cyclic then G contains two different elements g_1 and g_2 such that $G = \langle g_1 \rangle = \langle g_2 \rangle$.
10. If G is cyclic of order 9 then G contains six different elements g_1, g_2,..., g_6 such that $G = \langle g_1 \rangle = \cdots = \langle g_6 \rangle$.
11. If $G = GL(2, \mathbb{R})$, then some elements of G have finite order and some have infinite order.
12. $\mathbb{Z}_7^*$ is a cyclic group.
13. Every group of size 4 is abelian.

2. Let D_8 be the dihedral group of size 8 consisting of the rotations e, ρ, ρ^2, ρ^3 and reflections $\sigma_1, \ldots \sigma_4$ described in lectures. Write $\sigma = \sigma_1$. Prove

(a) $\sigma\rho = \rho^{-1}\sigma$

(b) $\{\sigma, \rho\sigma, \rho^2\sigma, \rho^3\sigma\} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$

(c) for any i, j, the product $\sigma_i\sigma_j$ is a rotation (*hint: use (a) and (b)*)

(d) D_8 has five elements of order 2 and two elements of order 4

(e) D_8 has exactly seven different cyclic subgroups.

3. Do Q2, parts (a)–(c) for the dihedral group D_{2n} for n an arbitrary integer with $n \geq 3$.

4. Let Π be the infinite strip pattern

...D D D D D...

Show that every element of the symmetry group $G(\Pi)$ is of the form τ^n or $\tau^n\sigma$, where τ is a suitable translation and σ is a suitable reflection. Prove that $G(\Pi)$ is abelian.

5. For each of the following figures, describe the elements of the symmetry group of the figure, and state which of the groups is abelian:

(a) A non-square rectangle

(b) A circle.

(c) Two hexagons stuck together along one edge.

(d) A pacman (2d, without eyes).

(e) $\mathbb{Z}^2$.

6.‡ Does there exist a group G, a subgroup H, and an element $g \in G$ such that $gH \subseteq Hg$ but $gH \neq Hg$? In other words, can a right coset be strictly bigger than a left coset?