KMB, 09/12/08

L-functions, Problem Sheet 4

[note: sign error in Q3d now fixed.]

This sheet finishes the proof of the meromorphic continuation of the local zeta functions, by checking the meromorphic continuation of the $\rho(c)$ associated to one explicit function f, in each of the many cases that we have to do. It also does some other bits and bobs related to this sort of stuff.

1)

a) Prove that any continuous group homomorphism $\mathbf{R} \to \mathbf{C}^\times$ is of the form $x \mapsto e^{sx}$ for some complex number s [hint: for an arbitrary continuous group homomorphism, consider a tiny open neighbourhood of 1 in $\mathbf{C}^\times$ on which there is a holomorphic branch of the logarithm sending 1 to 0; now consider $\mathbf{Q} \subseteq \mathbf{R}$ in a neighbourhood of zero, where linearity tells you a lot, and then extend to $\mathbf{R}$ using continuity].

b) Prove from first principles that the Pontrjagin dual of $\mathbf{R}$ is $\mathbf{R}$. The "problem" with the proof given in the course was that, when showing that $i(\mathbf{R})$ was dense in $\widehat{\mathbf{R}}$, I needed a duality between closed subgroups of $\mathbf{R}$ and closed subgroups of its dual, which as far as I can see is a "deep statement". Fix this up by proving $i(\mathbf{R})$ is dense in $\widehat{\mathbf{R}}$ from first principles.

2) Give an example of a continuous injection $i : X \to Y$ from a complete metric space into another complete metric space, such that i is a homeomorphism onto its image, and a sequence $(x_n)_{n\geq 1}$ in X with the property that (x_n) does not converge but $i(x_n)$ does. This is why I had to be very careful when proving that the map $i : K \to \hat{K}$ in the course had closed image.

In the remaining questions we go through some of the $\rho(c)$ calculations that I skipped in the course. Everything is elementary but the calculations in full generality are quite long, so I decided to skip them. I do all the cases here and I have just copied it, hopefully correctly, from Tate, except that I've added the $K = \mathbf{Q}_p$ case as a hopefully easier special case of the general p-adic case. Note also that Q5 and Q6 are relatively painless, but that Q4 is quite long.

3) Let's do the part of $K = \mathbf{Q}_p$ which I didn't do in the lecture, that is, let's go through the details of the meromorphic continuation of $\rho(c)$ in the ramified component case. Notation: we write $\mathbf{Q}_p^\times = U \times V$ with U the units and V the infinite cyclic group generated by p. We fix a Dirichlet character χ of conductor p^n, $n \geq 1$, and consider the component of $Q = \mathrm{Hom}(\mathbf{Q}_p^\times, \mathbf{C}^\times)$ consisting of characters of the form $p^j u \mapsto \chi(u)p^{-js}$. We let f be the function sending x to 0 for $|x| > p^n$ and to $e^{2\pi i q(x)}$ for $|x| \leq p^n$. We checked that $\hat{f} = p^n \chi_{1+p^n\mathbf{Z}_p}$ and that $f \in Z$. Let's finish the job from here.

a) We first need to compute $\zeta(f, c)$ for c in the component above and $0 < \mathrm{Re}(c) < 1$. First check that if $c(p^j u) = \chi(u)p^{-js}$ then (writing $x = p^j u$) we have

$$\zeta(f,c) = \sum_{j=-n}^{\infty} p^{-js} \frac{p}{p-1} \int_{u \in \mathbf{Z}_p^\times} e^{2\pi i q(up^j)} \chi(u) \mathrm{d}\mu(x).$$

b) Check that for $j \geq 0$ this integral vanishes [hint: χ is non-trivial].

c) Check that in fact for $-n < j < 0$ the integral also vanishes. Hint: write $\mathbf{Z}_p^\times = \cup_\alpha \alpha + p^{-j}\mathbf{Z}_p$, where α runs through a set of coset representatives for $(\mathbf{Z}/p^{-j}\mathbf{Z})$, the point being that $q(up^j)$ is constant on these cosets, and check that $\int_{u\in\alpha+p^{-j}\mathbf{Z}_p} \chi(u)\mathrm{d}\mu(u)$ vanishes, using the fact that χ is primitive (this is the only point in the calculation where it's used).

d) Deduce that $\zeta(f,c) = \frac{p^{ns-n+1}}{p-1}\sum_{\alpha\in(\mathbf{Z}/p^n\mathbf{Z})^\times}\zeta^\alpha\chi(\alpha)$ where $\zeta = e^{2\pi i/p^n}$.

e) [much easier!] Check that $\zeta(\hat{f},\hat{c}) = \frac{p}{p-1}$.

f) Deduce that $\rho(c) = \zeta(f,c)/\zeta(\hat{f},\hat{c})$ has meromorphic continuation to all $s \in \mathbf{C}$.

4) [only for the people who want to see everything]. Check that $\rho(c)$ has a meromorphic continuation for K a finite extension of $\mathbf{Q}_p$. Here's which functions to use. Let J denote the "inverse different" in K, that is, $J = \{x \in K : \mathrm{Tr}_{K/\mathbf{Q}_p}(xy) \in \mathbf{Z}_p \forall y \in \mathbf{Z}_p\}$. Let ϖ denote a fixed uniformiser and let V be the subgroup of $K^\times$ generated by ϖ. Recall that $J^{-1} = (\varpi^r)$ for some $r \geq 0$ and $N_{K/\mathbf{Q}}(\varpi^r) = p^m u$ for some unit u, where p^m is the discriminant of $K/\mathbf{Q}_p$.

a) Let's start with the component corresponding to the trivial character of U. Let f be the characteristic function of J. Check that this is in Z, and that for $0 < \mathrm{Re}(s) < 1$ we have that $\zeta(f,|.|^s) = p^{m(s-\frac{1}{2})}/(1-q^{-s})$ and that $\zeta(\hat{f},|.|^{1-s}) = 1/(1-q^{s-1})$. Deduce that $\rho(|.|^s)$ has a meromorphic continuation in this case. Deduce also that, with our choice of identification of K with $\hat{K}$ and our choice of Haar measure on K, the constant in the Fourier inversion theorem is 1, that is, $\hat{\hat{g}}(x) = g(-x)$ for any g.

b) Now say $\chi : U \to S^1$ is non-trivial. Choose $n \geq 1$ minimal such that $1 + \varpi^n R \subseteq \ker(\chi)$ with R the integers of K. Let f be zero away from $J.\varpi^{-n}$ and be $e^{2\pi i\Lambda(x)}$ on $J.\varpi^{-n}$. Mimic the proof in the last question to deduce again that $\rho(c)$ has meromorphic continuation for c in the component corresponding to χ. A crib is in Tate.

Let's now prove the meromorphic continuation of ρ in the real and complex cases. First let's establish the basic integrals we need.

5)

a) Prove that $\int_{-\infty}^{\infty} e^{-\pi x^2}\mathrm{d}x = 1$. [hint: if the integral is I then compute the double integral I^2 by switching to polar coordinates.]

b) Deduce that $\int_{-\infty}^{\infty} e^{-\pi x^2+2\pi ixy}\mathrm{d}x = e^{-\pi y^2}$ (hint: complete the square and use Cauchy).

c) Deduce that $\int_{-\infty}^{\infty} xe^{-\pi x^2+2\pi ixy}\mathrm{d}x = iye^{-\pi y^2}$. Hint: use (b) and differentiate under the integral.

6) Now let's nail the case $K = \mathbf{R}$.

a) For the trivial component, set $f(x) = e^{-\pi x^2}$. Check that $f \in L^1(\mathbf{R})$, that $f(x)|x|^\sigma \in L^1(\mathbf{R}^\times)$ for $\sigma > 0$, and that $\hat{f} = f$. Deduce $f \in Z$, amd furthermore that the constant in the Fourier inversion theorem is 1 with our choice of normalisations. Check that for $0 < \mathrm{Re}(s) < 1$ we have $\zeta(f,|.|^s) = \pi^{-s/2}\Gamma(s/2)$ and $\zeta(\hat{f},|.|^{1-s}) = \pi^{(s-1)/2}\Gamma((1-s)/2)$. Deduce that $\rho(|.|^s)$ has meromorphic continuation to $s \in \mathbf{C}$. If you can remember Legendre's Duplication Formula and Euler's Reflection Formula (neither of which we proved), then show $\rho(|.|^s) = 2^{1-s}\pi^{-s}\cos(\pi s/2)\Gamma(s)$.

b) For the other component, set $f(x) = xe^{-\pi x^2}$. Again check that $f \in L^1(\mathbf{R})$ and $f(x)|x|^\sigma \in L^1(\mathbf{R}^\times)$ for $\sigma > 0$. Check that $\hat{f} = if$ and deduce $f \in Z$. Show that for $0 < \mathrm{Re}(s) < 1$ we have $\zeta(f, \mathrm{sgn}(\cdot).|\cdot|^s) = \pi^{(-s-1)/2}\Gamma((s+1)/2)$ and $\zeta(\hat{f}, \mathrm{sgn}(\cdot).|\cdot|^{1-s}) = i\pi^{(s-2)/2}\Gamma((2-s)/2)$. Deduce that $\rho(, \mathrm{sgn}(\cdot).|\cdot|^s)$ has meromorphic continuation to all $s \in \mathbf{C}$ and, assuming Legendre's Duplication Formula and Euler's Reflection Formula, that in fact it's $-i2^{1-s}\pi^{-s}\sin(\pi s/2)\Gamma(s)$.

7) Finally let's do $K = \mathbf{C}$.

A character of $U = S^1$ is of the form $z \mapsto z^n$ for some $n \in \mathbf{Z}$. If $n \geq 0$ set $f_n(x+iy) = (x-iy)^n e^{-2\pi(x^2+y^2)}$ and if $n \leq 0$ set $f_n(x+iy) = (x+iy)^{-n}e^{-2\pi(x^2+y^2)}$. We'll use f_n on the component corresponding to the character $z \mapsto z^n$.

First check that $f_n \in L^1(\mathbf{C})$ and $f_n(z).|z|^\sigma \in L^1(\mathbf{C}^\times)$.

Next, we want to prove that $\hat{f}_n(z) = i^{|n|}f_{-n}(z)$ for all $n \in \mathbf{Z}$. Prove this as follows.

a) Check it for $n = 0$. Deduce that $f_0 \in Z$ and that the constant is 1 in the Fourier Inversion Theorem while we're at it.

b) By differentiating under the integral sign, prove it for all $n \geq 0$ by induction.

c) Using Fourier inversion, deduce it for $n < 0$.

d) Deduce $f_n \in Z$ for all $n \in \mathbf{Z}$.

e) Check that $\zeta(f_n, re^{i\theta} \mapsto r^{2s}e^{in\theta}) = (2\pi)^{1-s+\frac{|n|}{2}}\Gamma(s+\frac{|n|}{2})$ if $0 < \mathrm{Re}(s) < 1$.

f) Check that $\zeta(\hat{f}_n, \hat{c})$ (for c the character in part (e)) is $i^{|n|}(2\pi)^{s+\frac{|n|}{2}}\Gamma(1-s+\frac{|n|}{2})$. Deduce that again $\rho(c)$ has a meromorphic continuation to the component containing c.