KMB, 14/11/08

L-functions, Problem Sheet 3

Some of these exercises were already alluded to in the course. This is all Haar measurey stuff.

1) If $\hat{f}$ denotes the usual Fourier transform of the function $f : \mathbf{R} \to \mathbf{C}$ then check

a) If $g(x) = f(x+r)$ (r real) then $\hat{g}(y) = \hat{f}(y)e^{iry}$.

b) If $g(x) = f(x)e^{i\lambda x}$ for $\lambda \in \mathbf{R}$ then $\hat{g}(y) = \hat{f}(y-\lambda)$.

2) Check that if G is the integers and we define the open sets to be $[n, \infty) \cap \mathbf{Z}$ for all n, and the empty set and the whole space, then multiplication is continuous on G but inversion isn't.

3)

a) Check that if K is a normed field, then the map $K^\times \to K^2$ sending u to (u, u^{-1}) induces a homeomorphism between $K^\times$ (with the subspace topology induced from K) and its image (with the subspace topology of the product topology).

b) A topological ring is a ring R with a topology such that $(R, +)$ is a topological group, and multiplication is continuous. Can you find a topological ring with the property that the map $R^\times \to R^2$ analogous to that in part (a) is not a homeomorphism onto its image?

4) Let G be a topological group. Prove that $\{e\}$ is closed iff G is Hausdorff.

5) Prove that if X is the topological space $\mathbf{Q}$ with the subspace topology induced from $\mathbf{R}$, and $f : \mathbf{Q} \to \mathbf{R}$ is continuous with compact support, then $f = 0$.

6) G a LCHTG and notation as in the course. Say $f, g, h, F \in \mathcal{K}_+(G)$. We used both (a) and (b) below in the course when proving existence and uniqueness of Haar measure.

a) Prove $(f : g)(g : h) \geq (f : h)$

b) Prove $(\eta : f)^{-1} \leq \mu_F(f) \leq (f : \eta)$.

7) Check that the matrices $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{GL}_2(\mathbf{R})$ form a locally compact topological group in which the right Haar integral isn't a left Haar integral.

8) Notation: G is a LCHTG and we fix a (right) Haar integral μ on G. If $f \in \mathcal{K}(G)$ and $x \in G$ then define ${}^x f \in \mathcal{K}(G)$ by ${}^x f(g) = f(x^{-1}g)$.

a) Check that, for fixed $x \in G$, the functional $f \mapsto \mu({}^x f)$ is a Haar integral, and deduce that there's some $\Delta(x) \in \mathbf{R}_{>0}$ such that $\mu({}^x f) = \Delta(x)\mu(f)$ for all f.

b) Check that Δ is independent of the choice of μ. We say that Δ is the *modular function* associated to G.

c) Check that Δ is a continuous group homomophism $G \to \mathbf{R}$.

d) Check that Δ is the trivial homomorphism (that is, the map sending all $g \in G$ to 1) iff all right Haar integrals on G are also left Haar integrals. We say that a group G is *unimodular* if Δ is trivial.

e) Deduce the perhaps surprising result that compact LCHTGs are unimodular.

f) Compute Δ explicitly for the group in Q7.