KMB, 12/11/08

L-functions, Problem Sheet 2

Some of these exercises were already alluded to in the course. It's all p-adic stuff.

1) Use Hensel's Lemma to check that -7 is a square in $\mathbf{Q}_2$ and that $1-p$ is a square in $\mathbf{Q}_p$ for $p>2$. Deduce that $\mathbf{Q}$ isn't complete with respect to any p-adic norm.

2) Check the following assertion (used several times in the lectures): if k is a normed field and L is a complete normed field and $k \to L$ is a norm-preserving injection and a field homomorphism, then the closure of k in L is the completion of k.

3) If K is field with a non-arch norm $|.|$, then let's define a function $||.||$ on the polynomial ring $k[X]$ by $||\sum_{i=0}^n a_iX^i|| = \max_i |a_i|$.

Now say $g \in K[X]$ is non-zero, and $g = \sum_{i=0}^m a_iX^i$ with $||g|| = |a_m|$ (that is, no coefficient is bigger than the leading coefficient). Check that if $f \in K[X]$ is arbitrary and we write $f = qg + r$ (q for quotient, r for remainder) with $\deg(r) < n$, then we have $||q||.||g|| \leq ||f||$ and $||r|| \leq ||f||$.

4) Now say K is complete with respect to a non-arch norm. Say $f = \sum_{i=0}^n a_iX^i \in K[X]$ has degree $n \geq 2$, and suppose there exists an integer $0 < m < n$ such that $||f|| = |a_m|$ (see Q3 for notation) and $|a_i| < |a_m|$ for all $m < i \leq n$. In this question we'll use a Hensel's Lemma-like technique to prove that f cannot be irreducible in $K[X]$.

(a) Define $g_1 = \sum_{i=0}^m a_iX^i$ and $h_1 = 1$. Define δ by $||f - g_1h_1|| = \delta||f||$. Check $0 < \delta < 1$.

(b) Now say $t \geq 1$ and we have defined g_t, h_t with $\deg(g_t) = m$, $\deg(h_t) \leq n - m$, $||f - g_t|| \leq \delta||f||$, $||h_t - 1|| \leq \delta$, and $||f - g_th_t|| \leq \delta^t||f||$. Apply Q3 to the polynomial $f' := f - g_th_t$ and $g' := g_t$ (check that the assumptions apply). We get we get $f - g_th_t = qg_t + r$, and define $g_{t+1} = g_t + r$ and $h_{t+1} = h_t + q$. Check that g_{t+1} and h_{t+1} satisfy the assumptions we started with but with $t+1$ replacing t.

(c) Check that g_t and h_t tend coefficientwise to polynomials g and h with $gh = f$ and $\deg(g) = m$.

In the next couple of questions we'll prove the lemma we used in the course about unique extensions of norms for finite extensions of complete non-archimedean fields.

5) Say $|.|_1$ and $|.|_2$ are two norms on a field k. Suppose that $|.|_1$ is non-trivial, and that if $|a|_1 < 1$ then $|a|_2 < 1$. In this question we'll deduce that $|.|_1$ and $|.|_2$ are equivalent norms.

a) Check that $|a|_1 > 1$ implies $|a|_2 > 1$.

b) Show that $|a|_1 = 1$ implies $|a|_2 = 1$ (hint: $|.|_1$ is non-trivial; choose $c \in k^\times$ with $|c|_1 \neq 1$ and consider ca^n for appropriate $n \in \mathbf{Z}$).

c) Deduce that $|a|_1 < 1$ iff $|a|_2 < 1$ and similarly for $= 1$ and > 1.

d) Now for $c \in k^\times$ with $|c|_1 \neq 1$, and for arbitrary $b \in k^\times$, apply part (c) to b^nc^m for all integers m, n to deduce that if $|c|_1^\lambda = |c|_2$ then $|b|_1^\lambda = |b|_2$, which proves the result.

6) Show that if two norms on a field k induce the same topology, then they are equivalent (hint: use the previous question and the observation that $|x| < 1$ iff $(x^n)_{n \geq 1} \to 0$ in a normed field).

7) (norms on vector spaces) Let $(k, |.|)$ be a normed field, and suppose that the norm satisfies the triangle inequality. Let V be a vector space over k, equipped with a function $||.|| : V \to \mathbf{R}_{\geq 0}$. We say that V is a *normed vector space* over k if

(i) $||v|| = 0$ iff $v = 0$

(ii) $||v + w|| \leq ||v|| + ||w||$

(iii) $||\lambda v|| = |\lambda|.||v||$.

Two norms $||.||_1$ and $||.||_2$ on V are *equivalent* if there exists positive constants c and C with $c||v||_1 \leq ||v||_2 \leq C||v||_1$ for all $v \in V$. In this question we'll prove that if k is complete and V is finite-dimensional, then any two norms on V are equivalent, and furthermore that, in this situation, the distance function $d(v, w) = ||v - w||$ on V induced by any norm makes V a complete metric space.

(a) We're assuming k is complete and V finite-dimensional. Choose a basis $(e_1, e_2, \ldots, e_n)$ for V and define $||.||_0$ on V by $||\sum_i \lambda_i e_i||_0 = \max_i |\lambda_i|$. Show that this is a norm, that V is complete with respect to this norm, and hence that to finish the job all we have to do is to show that any norm on V is equivalent to $||.||_0$.

(b) Say $||.||$ is any norm on V. Show that if $C = \sum_j ||e_j||$ then $||v|| \leq C||v||_0$.

(c) We prove the other inequality by induction on n. Assume all norms on an $n-1$-dimensional space are equivalent, and make the space complete (this is true for $n = 1$ and $n = 2$). We proceed by contradiction. Suppose there exists no constant $c > 0$ such that $c||v||_0 \leq ||v||$. Deduce that for some $1 \leq i \leq n$, there exists a sequence $w_1, w_2, \ldots$ of vectors in V_i, the linear span of e_j, $j \neq i$, with the property that $||w_t + e_i|| \to 0$ as $t \to \infty$. Show that w_t converges in V_i (with respect to either induced norm on V_i—the inductive hypothesis shows they're equivalent), to some $w \in V_i$. Check $||w + e_i|| = 0$ and observe that this is a contradiction.

Now say L is a finite extension of the complete normed field k.

8) Use questions 6 and 7 to deduce that there is at most one extension of the norm on k to L, and that it makes L complete.

9) Prove that if $(k, |.|)$ is furthermore non-archimedean (this assumption is unnecessary but makes life a bit easier) then there does exist an extension $||.||$ of $|.|$ to L, namely $||\lambda|| = |N_{L/k}(\lambda)|^{1/n}$, with n the degree of L/k and N the norm map. Hint: The only tough part is to show $||a|| \leq 1$ implies $||1 + a|| \leq 1$. Let F be the characteristic polynomial of a (considered as an endomorphism of the k-vector space L). General field theory shows that $F = f^d$ where f is the minimal polynomial of a (and in particular f is irreducible). Then f is monic; check its constant term is an integer in k. If all the coefficients of f are integers then check we're done; if a coefficient of f isn't integral then use Q4 to get a contradiction.