SLOPES OF MODULAR FORMS

KEVIN BUZZARD AND TOBY GEE

Abstract. We survey the progress (or lack thereof!) that has been made on some questions about the p-adic slopes of modular forms that were raised by the first author in [Buz05], discuss strategies for making further progress, and examine other related questions.

1. Introduction

1.1. The question of the distribution of the local components of automorphic representations at finite places has received a great deal of attention.

In the case of fixing an automorphic representation and varying the finite place, we now have the recently-proved Sato–Tate conjecture for elliptic curves over totally real fields [HSBT10, CHT08, Tay08]. More recently, there has been much progress on questions where the automorphic representation varies, but the finite place is fixed; see [Shi12], and the references discussed in its introduction, for a detailed history of the question. Still more recently, there has been the fascinating work of [ST12] on hybrid problems, where both the finite place and the automorphic representation are allowed to vary, but we will have nothing to say about this here.

In this survey we will consider some other variants of this basic question, including p-adic ones. Just as in the classical setting, there are really several questions here, which will have different answers depending on what is varying: for example if one fixes a weight 2 modular form corresponding to a non-CM elliptic curve, then it is ordinary for a density one set of primes; however if one fixes a prime and a level and considers eigenforms of all weights, then almost none of them are ordinary (the dimension of the ordinary part remains bounded by Hida theory as the weight gets bigger).

We will for the most part limit ourselves to the case of classical modular forms for several reasons. The questions we consider are already interesting (and largely completely open) in this case, and in addition, there appear to be interesting phenomena that we do not expect to generalise in any obvious way (see Remark 4.1.9 below.) However, it seems worth recording a natural question (from the point of view of the p-adic Langlands program) about the distribution of local parameters as the tame level varies; for concreteness, we phrase the question for GL_n over a CM field, but the same question could be asked in greater generality in an obvious fashion.

Fix a CM field F, and consider regular algebraic essentially conjugate self-dual cuspidal automorphic representations π of GL_n/F. Fix an isomorphism between

Both authors were supported in part by EPSRC grant EP/L025485/1. The second author was additionally supported by a Leverhulme Prize, Marie Curie Career Integration Grant 303605, and ERC Starting Grant 306326.

$\overline{\mathbb{Q}}_p$ and $\mathbb{C}$, and a place $v|p$ of F. Assume that π_v is unramified (one could instead consider π_v lying on a particular Bernstein component). To such a π is associated a Galois representation $\rho_\pi : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p)$, and $\rho_\pi|_{\mathrm{Gal}(\overline{F_v}/F_v)}$ is crystalline, with Hodge–Tate weights determined by π_∞. (See the introduction to [CH13] for this result, and a discussion of the history of its proof. Thanks to the work of [HLTT13] and [Var14], the result is now known without the assumption of essentially conjugate self-duality; but the cuspidal automorphic representations of a fixed regular algebraic infinite type which are *not* essentially conjugate self-dual are expected to be rather sparse, and in particular precise asymptotics for the number of such representations as the level varies are unknown, and it therefore seems unwise to speculate about equidistribution questions for them. Note that in the essentially conjugate self-dual case, these automorphic representations arise via endoscopy from automorphic representations on unitary groups which are discrete series at infinity, and can thus be counted by the trace formula.) If we now run over π' of the same infinity type, which have π'_v unramified, and which furthermore have $\overline{\rho}_{\pi'} \cong \overline{\rho}_\pi$ (the bar denoting reduction to $\mathrm{GL}_n(\overline{\mathbb{F}}_p)$), then the local representations $\rho_{\pi'}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ naturally give rise to points of the corresponding (framed) deformation ring for crystalline lifts of $\overline{\rho}_\pi|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ of the given Hodge–Tate weights. The existence of level-raising congruences mean that one can often prove that this (multi)set is infinite (and it is expected to always be infinite), and one could ask whether some form of equidistribution of the $\rho_{\pi'}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$ holds in the rigid-analytic generic fibre of the crystalline deformation ring.

Unfortunately, this appears to be a very hard problem. Indeed, we do not in general even know that every irreducible component of the generic fibre of the local deformation space contains even a single $\rho_{\pi'}|_{\mathrm{Gal}(\overline{F}_v/F_v)}$; it is certainly expected that this holds, and a positive solution would yield a huge improvement on the existing automorphy lifting theorems (*cf.* the introduction to [CEG$^+$13]). Automorphy lifting theorems can sometimes be used to show that if an irreducible component contains an automorphic point, then it contains a Zariski-dense set of automorphic points, but they do not appear to be able to say anything about p-adic density, or about possible equidistribution.

More generally, one could allow the weight (and, if one wishes, the level at p) to vary (as well as, or instead of, allowing the level to vary) and ask about equidistribution in the generic fibre of the full deformation ring, with no p-adic Hodge theoretic conditions imposed. The points arising will necessarily lie on the sublocus of crystalline (or more generally, if the level at p varies, potentially semistable) representations, but as these are expected to be Zariski dense (indeed, this is known in most cases by the results of [Che13] and [Nak14]), it seems reasonable to conjecture that the points will also be Zariski dense.

One could also consider the case of a place $v \nmid p$, where very similar questions could be asked (except that there are no longer any p-adic Hodge-theoretic conditions), and we are similarly ignorant (although the automorphy lifting machinery can often be used to show that each irreducible component contains an automorphic point, using the Khare–Wintenberger method [KW09, Thm. 3.3] and Taylor's Ihara-avoidance result [Tay08]; see [Gee11, §5]).

In the case of modular forms (over $\mathbb{Q}$) one can make all of this rather more concrete, due to a pleasing low-dimensional coincidence: an irreducible two-dimensional crystalline representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is almost always completely determined by its Hodge–Tate weights and the trace of the crystalline Frobenius (because there is almost always a unique weakly admissible filtration on the associated filtered ϕ-module – see Section 4.1 below). This means that if we work with modular forms of weight k and level prime to p, the local p-adic Galois representation is almost always determined by the Hecke eigenvalue a_p (modulo the issue of semisimplicity in the ordinary case), and the question above reduces to the question of studying the p-adic behaviour of a_p. Such questions were studied computationally (and independently) by Gouvêa and one of us (KB), for the most part in level 1, when the era of computation of modular forms was in its infancy. Gouvêa noticed (see the questions in §2 of [Gou01]) that in weight k, the p-adic valuation $v_p(a_p)$ of a_p (normalised so that $v_p(p) = 1$) was almost always at most $(k-1)/(p+1)$, an observation which at the time did not appear to be predicted by any conjectures. Gouvêa and Buzzard also noticed that $v_p(a_p)$ was almost always an integer, an observation which even now is not particularly well-understood. Furthermore, in level 1, the primes p for which there existed forms with $v_p(a_p) > (k-1)/(p+1)$ seemed to *coincide* with the primes for which there existed forms with $v_p(a_p) \notin \mathbb{Z}$. These led Buzzard in §1 of [Buz05] to formulate the notion of an $\mathrm{SL}_2(\mathbb{Z})$-irregular prime, a prime for which there exists a level 1 non-ordinary eigenform of weight at most $p+1$. Indeed one might even wonder whether the following are equivalent:

- p is $\mathrm{SL}_2(\mathbb{Z})$-irregular;
- There exists a level 1 eigenform with $v_p(a_p) \notin \mathbb{Z}$;
- There exists a level 1 eigenform of weight k with $v_p(a_p) > (k-1)/(p+1)$.

One can check whether a given prime p is $\mathrm{SL}_2(\mathbb{Z})$-regular or not in finite time (one just needs to compute the determinant of the action of T_p on level 1 modular forms of weight k for each $k \leq p+1$ and check if it is always a p-adic unit; in fact one only has to check cusp forms of weights $4 \leq k \leq (p+3)/2$ because of known results about θ-cyles); one can also verify with machine computations that the second or third conditions hold by exhibiting an explicit eigenform with the property in question. The authors do not know how to verify with machine computations that the second or third conditions fail; equivalently, how to prove for a given p either that all T_p-eigenvalues a_p of all level 1 forms of all weights have integral p-adic valuations, or that they all satisfy $v_p(a_p) \leq (k-1)/(p+1)$. In particular it is still logically possible that for *every* prime number there will be some level 1 eigenforms satifying $v_p(a_p) \notin \mathbb{Z}$ or $v_p(a_p) > (k-1)/(p+1)$. However this seems very unlikely – for example $p = 2$ is an $\mathrm{SL}_2(\mathbb{Z})$-regular prime, and the first author has computed $v_p(a_p)$ for $p = 2$ and for all $k \leq 2048$ and has found no examples where $v_2(a_2) \notin \mathbb{Z}$ or $v_2(a_2) > (k-1)/3$. Gouvea also made substantial calculations for all other $p < 100$ which add further weight to the idea that the conditions are equivalent.

There are precisely two $\mathrm{SL}_2(\mathbb{Z})$-irregular primes less than 100, namely 59 and 79, and it does not appear to be known whether there are infinitely many $\mathrm{SL}_2(\mathbb{Z})$-regular primes or whether there are infinitely many $\mathrm{SL}_2(\mathbb{Z})$-irregular primes. (However, Frank Calegari has given `https://galoisrepresentations.wordpress.com/2015/03/03/review-of-buzzard-gee/` an argument which shows that under standard conjectures about the existence of prime values of polynomials with rational coefficients, then there are infinitely many $\mathrm{SL}_2(\mathbb{Z})$-irregular primes.) Note that for $p = 59$ and $p = 79$ eigenforms with $v_p(a_p) \notin \mathbb{Z}$ and $v_p(a_p) > (k-1)/(p+1)$ do exist, but any given eigenform will typically satisfy at most one of these conditions, and we do not even know how to show that the second and third conditions are equivalent.

Buzzard conjectured that for an $\mathrm{SL}_2(\mathbb{Z})$-regular prime, $v_p(a_p)$ was integral for all level 1 eigenforms, and even conjectured an algorithm to compute these valuations in all weights. Similar conjectures were made at more general levels $N > 1$ prime to p, and indeed Buzzard formulated the notion of a $\Gamma_0(N)$-regular prime – for $p > 2$ this is a prime $p \nmid N$ such that all eigenforms of level $\Gamma_0(N)$ and weight at most $p+1$ are ordinary, although here one has to be a little more careful when $p = 2$ (and even for $p > 2$ some care needs to be taken when generalising this notion to $\Gamma_1(N)$ because allowing odd weights complicates the picture somewhat; see Remark 4.1.5.)

These observations of Buzzard and Gouvêa can be thought of as saying something about the behaviour of the Coleman–Mazur eigencurve near the centre of weight space. Results of Buzzard–Kilford [BK05], Roe [Roe14], Wan–Xiao–Zhang [WXZ14] and Liu–Wan–Xiao [LWX14] indicate that there is even more structure near the boundary of weight space; this structure translates into concrete assertions about $v_p(a_p)$ when a_p is the U_p-eigenvalue of a newform of level $\Gamma_1(Np^r)$ and character of conductor Mp^r for some $M \mid N$ coprime to p. We make precise conjectures in Section 4.2. On the other hand, perhaps these results are intimately related to the p-adic Hodge-theoretic coincidence alluded to above – that in this low-dimensional situation there is usually only one (up to isomorphism) weakly admissible filtration on the Weil–Deligne representation in question. In particular such structure might not be so easily found in a general unitary group eigenvariety.

Having formulated these conjectures, in Section 5 we discuss a potential approach to them via modularity lifting theorems.

1.2. **Acknowledgements.** We would like to thank Christophe Breuil, Frank Calegari and Matthew Emerton for many helpful conversations over the past 14 years about the subjects of this survey. We would like to thank Vytautas Paškūnas for a helpful conversation about the Breuil–Mézard conjecture, and Liang Xiao for explaining some aspects of the papers [WXZ14] and [LWX14] to us. We would like to thank Frank Calegari, Matthew Emerton and James Newton for their helpful comments on an earlier draft of this paper; in addition, we would like to thank Frank Calegari for refereeing the paper (`https://galoisrepresentations.wordpress.com/2015/03/03/review-of-buzzard-gee/`) and for making several helpful comments in doing so. The second author would like to thank the organisers of the Simons Symposium on Automorphic Forms and the Trace Formula for the invitation to speak at the symposium, and for the suggestion of turning the rough notes for the talk into this survey.

2. Limiting distributions of eigenvalues

In this section we briefly review some conjectures and questions about the limiting distributions of eigenvalues of Hecke operators in the p-adic context. These questions will not be the main focus of our discussions, but as they are perhaps the most natural analogues of the questions considered in [ST12], it seems worth recording them.

2.1. $\ell = p$**: Conjectures of Gouvêa.** The reference for this section is the paper [Gou01]. Fix a prime p, an integer $N \geq 1$ coprime to p, and consider the operator U_p on the spaces of classical modular forms $S_k(\Gamma_0(Np))$ for varying weights $k \geq 2$. The characteristic polynomial of U_p has integer coefficients so it makes sense to consider the slopes of the eigenvalues – by definition, these are the p-adic valuations of the eigenvalues considered as elements of $\overline{\mathbb{Q}}_p$. The eigenvalues themselves fall into two categories. The ones corresponding to eigenforms which are new at p (corresponding to Steinberg representations) have U_p-eigenvalues $\pm p^{(k-2)/2}$, and thus slope $(k-2)/2$. The other eigenvalues come in pairs, each pair being associated to an eigenvalue of T_p on $S_k(\Gamma_0(N))$, and if the T_p-eigenvalue is a_p (considered as an element of $\overline{\mathbb{Q}}_p$) then the corresponding two p-oldforms have eigenvalues given by the roots of $x^2 - a_p x + p^{k-1}$; so the slopes $\alpha, \beta \in [0, k-1]$ satisfy $\alpha + \beta = k - 1$. Note that $\min\{\alpha, \beta\} = \min\{v_p(a_p), \frac{k-1}{2}\}$ by the theory of the Newton polygon, and in particular if $v_p(a_p) < \frac{k-1}{2}$ then $v_p(a_p)$ can be read off from α and β.

Now consider the (multi-)set of slopes of p-oldforms, normalised by dividing by $k-1$ to lie in the range $[0, 1]$. More precisely we could consider the measure (a finite sum of point measures, normalised to have total mass 1) attached to this multiset in weight k. Let k tend to ∞ and consider how these measures vary. Is there a limiting measure?

Conjecture 2.1.1. *(Gouvêa) The slopes converge to the measure which is uniform on* $[0, \frac{1}{p+1}] \cup [\frac{p}{p+1}, 1]$ *and* 0 *elsewhere.*

This is supported by the computational evidence, which is particularly convincing in the $\Gamma_0(N)$-regular case. This conjecture implies that if a_p runs through the eigenvalues of T_p on $S_k(\Gamma_0(N))$ then we "usually" have $v(a_p) \leq (k-1)/(p+1)$. This appears to be the case, although the reasons why are not well understood. If $p > 2$ is $\Gamma_0(N)$-regular however, the (purely local – see below) main result of [BLZ04] shows that $v(a_p) \leq \lfloor (k-2)/(p-1) \rfloor$. One might hope that the main result of [BLZ04] could be strengthened to show that in fact $v(a_p) \leq \frac{k-1}{p+1}$; it seems likely that the required local statement is true, but Berger tells us that the proof in [BLZ04] does not seem to extend to this more general range. (This problem is carefully examined in Mathieu Vienney's unpublished PhD thesis.)

2.2. $\ell \neq p$**.** In the previous subsection we talked about the distribution of a_p, the eigenvalues of T_p on $S_k(\Gamma_0(N))$, considered as elements of $\overline{\mathbb{Q}}_p$. The Ramanujan bounds and the Sato–Tate conjecture give us information about the eigenvalues of T_p as elements of the complex numbers. What about the behaviour of the a_p as elements of $\overline{\mathbb{Q}}_l$ for $\ell \neq p$ prime? We have very little idea what to expect. In this short section we merely present a sample of some computational results concerning the even weaker question of the distribution of the reductions of the a_p as elements of $\overline{\mathbb{F}}_l$. In contrast to the previous section we here vary N and keep $k = 2$ fixed. More precisely, we fix distinct ℓ and p, and then loop over $N \geq 1$ coprime to ℓp and compute the eigenvalues $\overline{a}_p$ of T_p acting on $S_2(\Gamma_0(N); \overline{\mathbb{F}}_l)$. Here is a sample of the results with $p = 5$ and $\ell = 3$, looping over the first 5,533,155 newforms:

Size of $\mathbb{F}_3[\overline{a}_5]$	Number of newforms
3^1	80656
3^2	38738
3^3	35880
3^4	32968
3^5	35330
3^6	33372
3^7	34601
3^8	33896
3^9	35262
3^{10}	33600

The numbers in the second column of this the table are *not* decreasing, which is perhaps not what one might initially guess; Frank Calegari observed that this could perhaps be explained by observing that if you choose a random element of a finite field $\mathbb{F}_q$ then the field it generates over $\mathbb{F}_p$ might be strictly smaller than $\mathbb{F}_q$, and the heuristics are perhaps complicated by this.

3. The Gouvêa–Mazur Conjecture/Buzzard's Conjectures

3.1. Coleman theory (see Theorem D of [Col97]) tells us that for a fixed prime p and tame level N, there is a function $M(n)$ such that if $k_1, k_2 > n+1$ and $k_1 \equiv k_2 \pmod{p^{M(n)}(p-1)}$, then the sequences of slopes (with multiplicities) of classical modular forms of level Np and weights k_1, k_2 agree up to slope n. A more geometric way to think about this theorem is that given a point on the eigencurve of slope $\alpha \leq n$, there is a small neighbourhood of that point in the eigencurve, which maps in a finite manner down to a disc in weight space of some explicit radius $p^{-M(n)}$ and such that all the points in the neighbourhood have slope α. Gouvêa and Mazur [GM92] conjectured that we could take $M(n) = n$; for $n = 0$, this is a theorem of Hida (his ordinary families are finite over entire components of weight space). Wan [Wan98] deduced from Coleman's results that $M(n)$ could be taken to be quadratic (with the implicit constants depending on both p and N; as far as we know, it is still an open problem to obtain a quadratic bound independent of either p or N). However, Buzzard and Calegari [BC04] found an explicit counterexample to the conjecture that $M(n) = n$ always works.

On the other hand, Buzzard [Buz05] accumulated a lot of numerical evidence that whenever p is $\Gamma_0(N)$-regular, many (but not all) families of eigenforms seemed to have slopes which were locally equal to n on discs of size $p^{-L(n)}$ with $L(n)$ seemingly linear in $\log(n)$ – a much stronger bound than the Gouvêa–Mazur conjectures. For example if $p = 2$, $N = 1$ then the classical slopes at weight $k = 2^d$ (the largest of which is approximately $k/3$) seem to be an initial segment of the classical slopes at weight 2^{d+1}. For example, the 2-adic slopes in level 1 and weight $128 = 2^7$ are

$$3, 7, 13, 15, 17, 25, 29, 31, 33, 37$$

and the Gouvêa–Mazur conjectures would predict that the slopes which were at most 7 should show up in weight $256 = 2^8$. However in weight 256 the slopes are

$$3, 7, 13, 15, 17, 25, 29, 31, 33, 37, 47, 49, 51, \ldots$$

and more generally the slopes at weight equal to a power of 2 all seem to be initial segments of the infinite slope sequence on overconvergent 2-adic forms of weight 0, a sequence explicitly computed in Corollary 1 of [BC05]. In particular, if one were to restrict to $p = 2$, $N = 1$ and k a power of 2 then $M(n)$ can be conjecturally taken to be the base 2 logarithm of $3n$. Note also that the counterexamples at level $\Gamma_0(N)$ to the Gouvêa–Mazur conjecture in [BC04] were all $\Gamma_0(N)$-irregular. It may well be the case that the Gouvêa–Mazur conjectures are true at level $\Gamma_0(N)$ if one restricts to $\Gamma_0(N)$-regular primes – indeed the numerical examples above initially seem to lend credence to the hope that something an order of magnitude stronger than the Gouvêa–Mazur conjectures might be true in the $\Gamma_0(N)$-regular case. However life is not quite so easy – numerical evidence seems to indicate that near to a newform for $\Gamma_0(Np)$ on the eigencurve, the behaviour of slopes seems to be broadly speaking behaving in the same sort of way as predicted by the Gouvêa–Mazur conjectures. For example, again with $p = 2$ and $N = 1$, computer calculations give that the slopes in weight $38 + 2^8$ are

$$5, 8, 16, 18, 18, 20, 29, 32, 37, 40, 45, 50, 50, 56, 61, 64, 70, \ldots$$

whereas in weight $38 + 2^9$ they are

$$5, 8, 17, 18, 18, 19, 29, 32, 37, 40, 45, 50, 50, 56, 61, 64, 70, \ldots.$$

Again one sees evidence of something far stronger than the Gouvêa–Mazur conjecture going on (the Gouvêa–Mazur conjecture only predicts equality of slopes which are at most 8); however there seems to be a family which has slope 16 in weight $38 + 2^8$ and slope 17 in weight $38 + 2^9$. This family could well be passing through a classical newform of level $\Gamma_0(2)$ in weight 38, and newforms in weight 38 have slope $(38-2)/2 = 18$, so one sees that for just this one family $M(n)$ is behaving much more like something linear in n.

Staying in the $\Gamma_0(N)$-regular case, Buzzard found a lot of evidence for a far more precise conjecture than the Gouvêa–Mazur conjecture – one that gives a complete description of the slopes in the $\Gamma_0(N)$-regular case, in terms of a recursive algorithm, which is purely combinatorial in nature and uses nothing about modular forms at all. Then (see [Buz05, §3] for a more detailed discussion) the algorithm can for the most part be deduced from various heuristic assumptions about families of p-adic modular forms, for example the very strong "logarithmic" form of the Gouvêa–Mazur conjecture mentioned above, plus some heuristics about behaviour of slopes near newforms that seem hard to justify. Unfortunately, essentially nothing is known about these conjectures, even in the simplest case $N = 1$ and $p = 2$, where the slopes are all conjectured to be integers but even this is not known.

In fact it does not even seem to be known that the original form of the Gouvêa–Mazur conjecture (in the $\Gamma_0(N)$-regular case) is a consequence of Buzzard's conjectures; see [Buz05, Q. 4.11]. It would also be of interest to examine Buzzard's original data to try to formulate a precise conjecture about the best possible value of $M(n)$ in the $\Gamma_0(N)$-regular case. The following are combinatorial questions, and are presumably accessible.

Question 3.1.1. Say p is $\Gamma_0(N)$-regular.

(1) Does the Gouvêa–Mazur conjecture for (p, N), or perhaps something even stronger, follow from Buzzard's conjectures?
(2) Does Conjecture 2.1.1 follow from Buzzard's conjectures?

One immediate consequence of Buzzard's conjectures is that in the $\Gamma_0(N)$-regular case, all of the slopes should be integers. This can definitely fail in the $\Gamma_0(N)$-irregular case (and is a source of counterexamples to the Gouvêa–Mazur conjecture), and we suspect that understanding this phenomenon could be helpful in proving the full conjectures (see the discussion in Section 5 below). In Section 4.1 we will explain a purely local conjecture that would imply this integrality.

Note that Lisa Clay's PhD thesis also studies this problem and makes the observation that the combinatorial recipes seem to remain valid when restricting to the subset of eigenforms with a fixed mod p Galois representation which is reducible locally at p.

4. Local questions

4.1. **The centre of weight space.** In this section we discuss some purely local conjectures and questions about p-adic Galois representations that are motivated by the conjectures of Section 3. We briefly recall the relevant local Galois representations and their relationship to the global picture, referring the reader to the introduction to [BG09] for further details. If $k \geq 2$ and $a_p \in \overline{\mathbb{Q}}_p$ with $v(a_p) > 0$, then there is a two-dimensional crystalline representation V_{k,a_p} with Hodge–Tate weights $0, k-1$, with the property that the crystalline Frobenius of the corresponding weakly admissible module has characteristic polynomial $X^2 - a_pX + p^{k-1}$. Furthermore, if $a_p^2 \neq 4p^{k-1}$ then V_{k,a_p} is uniquely determined up to isomorphism. This is easily checked by directly computing the possible Hodge filtrations on the weakly admissible module; see for example [BB10, Prop. 2.4.5]. This is a low-dimensional coincidence however – a certain parameter space of flags is connected of dimension zero in this situation.

The relevance of this representation to the questions of Section 3 is that if $f \in S_k(\Gamma_0(N), \overline{\mathbb{Q}}_p)$ is an eigenform with $a_p^2 \neq 4p^{k-1}$ (which is expected to always hold; in the case $N = 1$ it holds by Theorem 1 of [Gou01], and the paper [CE98] proves that it holds for general N if $k = 2$, and for general k, N if one assumes the Tate conjecture) then $\rho_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)} \cong V_{k,a_p}$.

As explained in [Buz05, §1], $p > 2$ is $\Gamma_0(N)$-regular if and only if $\overline{\rho}_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible for every $f \in S_k(\Gamma_0(N))$ (and every $k \geq 2$). This suggests that the problem of determining when $\overline{V}_{k,a_p}$ (the reduction of V_{k,a_p} modulo p) is reducible could be relevant to the conjectures of Section 3. To this end, we have the following conjecture.

Conjecture 4.1.1. *If p is odd, k is even and $v(a_p) \notin \mathbb{Z}$ then $\overline{V}_{k,a_p}$ is irreducible.*

Remark 4.1.2. Any modular form of level $\Gamma_0(N)$ necessarily has even weight, and this conjecture would therefore imply for $p > 2$ that in the $\Gamma_0(N)$-regular case, all slopes are integral, as Buzzard's conjectures predict (see §3 above).

Remark 4.1.3. This conjecture is arguably "folklore" but seems to originate in emails between Breuil, Buzzard and Emerton in 2005.

Remark 4.1.4. The conjecture is of course false without the assumption that $v(a_p) \notin \mathbb{Z}$; indeed, if $v(a_p) = 0$ then we are in the ordinary case, and V_{k,a_p} is reducible (and so $\overline{V}_{k,a_p}$ is certainly reducible).

Remark 4.1.5. If k is allowed to be odd then the conjecture would be false – for global reasons! There are p-newforms of level $\Gamma_1(N) \cap \Gamma_0(p)$ and odd weight k_0, which automatically have slope $(k_0 - 2)/2 \notin \mathbb{Z}$, and in computational examples these forms give rise to both reducible and irreducible local mod p representations.

The corresponding local p-adic Galois representations are now semistable rather than crystalline, and depend on an additional parameter, the $\mathcal{L}$-*invariant*; the reduction of the Galois representation depends on this $\mathcal{L}$-invariant in a complicated fashion, see for example the calculations of [BM02]. Considering oldforms which are sufficiently p-adically close to such newforms (and these will exist by the theory of the eigencurve) produces examples of V_{k,a_p} with $v(a_p) = (k_0 - 2)/2$ and $\overline{V}_{k,a_p}$ reducible. If k and k_0 are close in weight space then k will also be odd.

The main result of [BG13] determines, for odd p, exactly for which a_p with $0 < v(a_p) < 1$ the representation $\overline{V}_{k,a_p}$ is irreducible; it is necessary that $k \equiv 3 \pmod{p-1}$, that $k \geq 2p+1$, and that $v(a_p) = 1/2$, and there are examples for all k satisfying these conditions.

Remark 4.1.6. If $p = 2$ then the conjecture is also false for the trivial reason that if $k \equiv 4 \bmod 6$ then $\overline{V}_{k,0}$ is reducible and hence $\overline{V}_{k,a}$ is reducible for $v(a)$ sufficiently large (whether or not it is integral) by the main result of [BLZ04]. In particular, the conjecture does not offer a local explanation for the global phenomenon that thousands of slopes of cusp forms have been computed for $N = 1$ and $p = 2$, and not a single non-integral one has been found (and the conjectures of [Buz05] predict that the slopes will all be integral).

Remark 4.1.7. Conjecture 4.1.1 is known if $v(a_p) \in (0,1)$, which is the main result of [BG09]. It is also known if $v(a_p) > \lfloor (k-2)/(p-1) \rfloor$, by the main result of [BLZ04]. In the case that $k \leq (p^2+1)/2$, it is expected to follow from work in progress of Yamashita and Yasuda.

The result of [BLZ04] is proved by constructing an explicit family of (ϕ, Γ)-modules which are p-adically close to the representation $V_{k,0}$. Since $V_{k,0}$ is induced from a Lubin–Tate character, it has irreducible reduction if k is not congruent to 1 modulo $p+1$, and in particular has irreducible reduction when $p > 2$ and k is even, which implies the result.

In contrast, the papers [BG09, BG13] use the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ to compute $\overline{V}_{k,a_p}$ more or less explicitly. Despite the simplicity of the calculations of [BG09], which had originally made us optimistic about the prospects of proving Conjecture 4.1.1 in general, it seems that when $v(a_p) > 1$ the calculations involved in computing $\overline{V}_{k,a_p}$ are very complicated, and without having some additional structural insight we are pessimistic that Conjecture 4.1.1 can be directly proved by this method.

In the light of the previous remark, we feel that it is unlikely that Conjecture 4.1.1 will be proved without some gaining some further understanding of why it should be true. We therefore regard the following question as important.

Question 4.1.8. Are there any local or global reasons that we should expect Conjecture 4.1.1 to hold, other than the computational evidence of the second author discussed in [Buz05]?

Remark 4.1.9. It seems unlikely that any analogue of Conjecture 4.1.1 will hold in a more general setting (i.e. for higher-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, or for representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$ of dimension > 1, where $F/\mathbb{Q}_p$ is a non-trivial extension). The reason for this is that there is no analogue of the fact that V_{k,a_p} is completely determined by k and a_p; in these more general settings, additional parameters are needed to describe the p-adic Hodge filtration, and it is highly likely that the reduction mod p of the crystalline Galois representations will depend on these parameters. (Indeed, as remarked above, this already happens for semistable 2-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.)

For this reason we are sceptical that there is any simple generalisation of the conjectures of Section 3, except to the case of Hilbert modular forms over a totally real field in which p splits completely. For example, Table 5 in [Loe08] and the comments below it show that non-integral slopes appear essentially immediately when one computes with $U(3)$.

4.2. **The boundary of weight space.** Perhaps surprisingly, near the boundary of weight space, the combinatorics of the eigencurve seem to become simpler. For example if $N = 1$ and $p = 2$ one can compare Corollary 1 of [BC05] (saying that in weight 0 all overconvergent slopes are determined by a complicated combinatorial formula) with Theorem B of [BK05] (saying that at the boundary of weight space the slopes form an arithmetic progression).

Now let f be a newform of weight $k \geq 2$ and level $\Gamma_1(Np^r)$, with $r \geq 2$, and with character whose p-part χ has conductor p^r. For simplicity, fix an isomorphism $\mathbb{C} = \overline{\mathbb{Q}}_p$. Say f has U_p-eigenvalue α. One checks that the associated smooth admissible representation of $\mathrm{GL}_2(\mathbb{Q}_p)$ attached to f must be principal series associated to two characters of $\mathbb{Q}_p^\times$, one unramified (and sending p to α) and the other of conductor p^r. Now say ρ_f is the p-adic Galois representation attached to f.

By local-global compatibility (the main theorem of [Sai97]), and the local Langlands correspondence, the F-semisimplified Weil–Deligne representation associated to ρ_f at p will be the direct sum of two characters, one unramified and the other of conductor p^r. Moreover, the p-adic Hodge-theoretic coincidence still holds: there is at most one possible weakly admissible filtration on this Weil–Deligne representation with jumps at 0 and $k-1$, by Proposition 2.4.5 of [BB10] (or by a direct calculation).

The resulting weakly admissible module depends only on k, α and χ, and so we may call its associated Galois representation $V_{k,\alpha,\chi}$; the local-global assertion is then that this is representation is the restriction of ρ_f to the absolute Galois group of $\mathbb{Q}_p$. Let $\overline{V}_{k,\alpha,\chi}$ denote the semisimplification of the mod p reduction of $V_{k,\alpha,\chi}$. We propose a conjecture which would go some way towards explaining the results of [BK05], [Roe14], [Kil08] and [KM12]. We write v_χ for the p-adic valuation v on $\overline{\mathbb{Q}}_p$ normalised so that the image of v_χ on $\mathbb{Q}_p(\chi)^\times$ is $\mathbb{Z}$ (so for $p > 2$ we have $v_\chi(p) = 1/(p-1)p^{r-2}$.)

Conjecture 4.2.1. If $v_\chi(\alpha) \notin \mathbb{Z}$ then $\overline{V}_{k,\alpha,\chi}$ is irreducible.

This is a local assertion so does not follow directly from the results in the global papers cited above. The four papers above prove that $v_\chi(\alpha) \in \mathbb{Z}$ if α is an eigenvalue of U_p on a space of modular forms of level 2^r, 3^r, 5^2 and 7^2 respectively; note that in all these cases, all the local mod p Galois representations which show up are reducible locally at p, for global reasons. In fact, slightly more is true in the special case $p = 2$ and $r = 2$: in this case $\mathbb{Q}_p(\chi) = \mathbb{Q}_2$ so the conjecture predicts that if $v(\alpha) \notin \mathbb{Z}$ then $\overline{V}_{k,\alpha,\chi}$ is irreducible; yet in [BK05] it is proved that eigenforms of odd weight, level 4, and character of conductor 4, all have slopes in $2\mathbb{Z}$.

It is furthermore expected that in the global setting the sequence of slopes is a finite union of arithmetic progressions; see [WXZ14, Conj. 1.1]. Indeed, a version of this statement (sufficiently close to the boundary of weight space, in the setting of the eigenvariety for a definite quaternion algebra with $p > 2$) is proved by Liu–Wan–Xiao in [LWX14].

5. A strategy to prove Buzzard's conjectures

5.1. The following strategy for attacking the conjectures of Section 3 was explained by the second author to the first author in 2005, and was the motivation for the research reported on in the papers [BG09, BG13] (which we had originally hoped would result in a proof of Conjecture 4.1.1).

Assume that $p > 2$, and fix a continuous odd, irreducible (and thus modular, by Serre's conjecture), representation $\overline{\rho} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$. Assume further that $\overline{\rho}$ satisfies the usual Taylor–Wiles condition that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_p))}$ is irreducible.

Let $R_k^{\mathrm{loc}}(\overline{\rho})$ be the (reduced and p-torsion free) universal framed deformation ring for lifts of $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ which are crystalline with Hodge–Tate weights $0, k-1$. This connects to the global setting via the following consequence of the results of [Kis09].

Proposition 5.1.1. *Maintain the assumptions and notation of the previous two paragraphs, so that $p > 2$, and $\overline{\rho} : \mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ is a continuous, odd, irreducible representation with $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}/\mathbb{Q}(\zeta_p))}$ irreducible.*

Let N be an integer not divisible by p such that $\overline{\rho}$ is modular of level $\Gamma_1(N)$. If $p = 3$, assume further that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is not a twist of the direct sum of the mod p cyclotomic character and the trivial character. Fix an irreducible component of $\mathrm{Spec}\, R_k^{\mathrm{loc}}(\overline{\rho})[1/p]$. Then there is a newform $f \in S_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ such that $\overline{\rho}_f \cong \overline{\rho}$, and $\rho_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ corresponds to a point of $\mathrm{Spec}\, R_k^{\mathrm{loc}}(\overline{\rho})[1/p]$ lying on our chosen component.

Proof. This follows almost immediately from the results of [Kis09], exactly as in the proof of [Cal12, Prop. 3.7]. (Note that the condition that f is a newform of level $\Gamma_1(N)$ can be expressed in terms of the conductor of ρ_f, and thus in terms of the components of the local deformation rings at primes dividing N.)

More precisely, this argument immediately gives the result in the case that $\overline{\rho}_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is not a twist of an extension of the trivial representation by the mod p cyclotomic character. However, this assumption on $\overline{\rho}_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is needed only in the proof of [Kis09, Cor. 2.2.17], where this assumption guarantees that the Breuil–Mézard conjecture holds for $\overline{\rho}_f|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ (indeed, the Breuil–Mézard conjecture is proved under this assumption in [Kis09]). The Breuil–Mézard conjecture is now known for $p > 2$, except in the case that $p = 3$ and $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is a twist of the direct sum of the mod p cyclotomic character and the trivial character, so the result follows. (The case that $p \geq 3$ and $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is a twist of a non-split extension of the trivial character by the mod p cyclotomic character is treated in [Paš15], and the case that $p > 3$ and $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is a twist of the direct sum of the mod p cyclotomic character and the trivial character is proved in [HT13].) $\square$

Suppose that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible, and that Conjecture 4.1.1 holds. Consider a_p as a rigid-analytic function on $\mathrm{Spec}\, R_k^{\mathrm{loc}}(\overline{\rho})[1/p]$; since $v(a_p) \in \mathbb{Z}$ by assumption, we see that $v(a_p)$ is in fact constant on connected (equivalently, irreducible) components of $\mathrm{Spec}\, R_k^{\mathrm{loc}}(\overline{\rho})[1/p]$.

Corollary 5.1.2. *Maintain the assumptions of Proposition 5.1.1, and assume further that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible. Assume Conjecture 4.1.1. Then the set of slopes (without multiplicities) of T_p on newforms $f \in S_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ with $\overline{\rho}_f \cong \overline{\rho}$ is determined purely by k and $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$; more precisely, it is the set of slopes of (the crystalline Frobenius of the Galois representations corresponding to) components of* $\operatorname{Spec} R_k^{\mathrm{loc}}(\overline{\rho})[1/p]$.

Proof. This is immediate from Proposition 5.1.1 (and the discussion in the preceding paragraph). □

Remark 5.1.3. The conclusion of Corollary 5.1.2 seems unlikely to hold if $\overline{\rho}$ is allowed to be (globally) reducible; for example, if $p = 2$, it is known that the slopes of all cusp forms for $\mathrm{SL}_2(\mathbb{Z})$ are at least 3, but there are local crystalline representations of slope 1 (for example the local 2-adic representation attached to the unique weight 6 level 3 cuspidal eigenform). We do not know if there is any reasonable "local to global principle" when $\overline{\rho}$ is reducible.

It would be very interesting to be able to have some control on the multiplicities with which slopes occur in $S_k(\Gamma_1(N), \overline{\mathbb{Q}}_p)$ (for example, to show that these multiplicities agree for two weights which are sufficiently p-adically close, as predicted by the Gouvêa–Mazur conjecture), but it is not clear to us how such results could be extracted from the modularity lifting machinery. If all the irreducible components of $R_k^{\mathrm{loc}}(\overline{\rho})$ were regular, it would presumably be possible to use the argument of Diamond [Dia97] to relate the multiplicities of the same slope in different weights, but we do not expect this to hold in any generality.

Not withstanding this difficulty, one could still hope to prove the conjectures of [Buz05] up to multiplicity. If Conjecture 4.1.1 were known, the main obstruction to doing this would be obtaining a strong local constancy result for slopes as k varies p-adically. More precisely, we would like to prove the following purely local conjecture for some function $M(n)$ as in §3 above.

Conjecture 5.1.4. *Let $\bar{r} : \mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$ be reducible. If $n \geq 0$ is an integer, and $k, k' \geq n+1$ have $k \equiv k' \pmod{(p-1)p^{M(n)}}$, then there is a crystalline lift of $\bar{r}$ with Hodge–Tate weights $0, k-1$ and slope n if and only if there is a crystalline lift of $\bar{r}$ with Hodge–Tate weights $0, k'-1$ and slope n.*

It might well be possible to prove a weak result in the direction of Conjecture 5.1.4 by the methods of [Ber12] (more precisely, to prove the conjecture with a much worse bound on $M(n)$ than would be needed for interesting applications to the conjectures of [Buz05], but without any assumption on the reducibility of $\bar{r}$).

Corollary 5.1.2 (which shows, granting as always Conjecture 4.1.1, that the set of slopes which occur globally is the same as the set of slopes that occur locally) shows that it would be enough to prove the global version of this statement, and it is possible that the methods of [Wan98] could allow one to deduce a local constancy result where the dependence on n in "sufficiently close" is quadratic in n. (Note that while it is not immediately clear how to adapt the methods of [Wan98] to allow $\overline{\rho}$ to be fixed, it seems plausible that the methods used to prove [WXZ14, Thm. D] will be able to do this.) Note again that the computations of [BC04] (which in particular disprove the original Gouvêa–Mazur conjecture) mean that we cannot expect to deduce Conjecture 5.1.4 (for an optimal function $M(n)$ of the kind suggested by Buzzard's conjectures) from any global result that does not use the hypothesis that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible.

However, it seems plausible to us that a weak local constancy result of this kind, also valid in the case that $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is irreducible, could be bootstrapped to give the required strong constancy, provided that Conjecture 4.1.1 is proved. The idea is as follows: under the assumption of Conjecture 4.1.1, $v(a_p)$ is constrained to be an integer when $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ is reducible. If one could prove a result (with no hypothesis on the reducibility of $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$) saying that if k, k' are sufficiently close in weight space, then the small slopes of crystalline lifts of $\overline{\rho}|_{\mathrm{Gal}\,(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ of Hodge–Tate weights $0, k-1$ and $0, k'-1$ are also close, then the fact that the slopes are constrained to be integers could then be used to deduce that the slopes are equal (because two integers which differ by less than 1 must be equal.)

References

[BB10] Laurent Berger and Christophe Breuil, *Sur quelques représentations potentiellement cristallines de* $\mathrm{GL}_2(\mathbf{Q}_p)$, Astérisque (2010), no. 330, 155–211.

[BC04] Kevin Buzzard and Frank Calegari, *A counterexample to the Gouvêa-Mazur conjecture*, C. R. Math. Acad. Sci. Paris **338** (2004), no. 10, 751–753.

[BC05] ———, *Slopes of overconvergent 2-adic modular forms*, Compos. Math. **141** (2005), no. 3, 591–604.

[Ber12] Laurent Berger, *Local constancy for the reduction* mod$p$ *of 2-dimensional crystalline representations*, Bull. Lond. Math. Soc. **44** (2012), no. 3, 451–459.

[BG09] Kevin Buzzard and Toby Gee, *Explicit reduction modulo p of certain two-dimensional crystalline representations*, Int. Math. Res. Not. IMRN (2009), no. 12, 2303–2317.

[BG13] ———, *Explicit reduction modulo p of certain 2-dimensional crystalline representations, II*, Bull. Lond. Math. Soc. **45** (2013), no. 4, 779–788.

[BK05] Kevin Buzzard and L. J. P. Kilford, *The 2-adic eigencurve at the boundary of weight space*, Compos. Math. **141** (2005), no. 3, 605–619.

[BLZ04] Laurent Berger, Hanfeng Li, and Hui June Zhu, *Construction of some families of 2-dimensional crystalline representations*, Math. Ann. **329** (2004), no. 2, 365–377.

[BM02] Christophe Breuil and Ariane Mézard, *Multiplicités modulaires et représentations de* $\mathrm{GL}_2(\mathbf{Z}_p)$ *et de* $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ *en* $l = p$, Duke Math. J. **115** (2002), no. 2, 205–310, With an appendix by Guy Henniart.

[Buz05] Kevin Buzzard, *Questions about slopes of modular forms*, Astérisque (2005), no. 298, 1–15, Automorphic forms. I.

[Cal12] Frank Calegari, *Even Galois representations and the Fontaine–Mazur conjecture. II*, J. Amer. Math. Soc. **25** (2012), no. 2, 533–554.

[CE98] Robert F. Coleman and Bas Edixhoven, *On the semi-simplicity of the* U_p*-operator on modular forms*, Math. Ann. **310** (1998), no. 1, 119–127.

[CEG+13] Ana Caraiani, Matthew Emerton, Toby Gee, David Geraghty, Vytautas Paškūnas, and Sug Woo Shin, *Patching and the p-adic local Langlands correspondence*, 2013.

[CH13] Gaëtan Chenevier and Michael Harris, *Construction of automorphic Galois representations, II*, Camb. J. Math. **1** (2013), no. 1, 53–73.

[Che13] Gaëtan Chenevier, *Sur la densité des représentations cristallines de Gal*$(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, Math. Ann. **355** (2013), no. 4, 1469–1525.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 1–181, With Appendix A, summarizing unpublished work of Russ Mann, and Appendix B by Marie-France Vignéras.

[Col97] Robert F. Coleman, *p-adic Banach spaces and families of modular forms*, Invent. Math. **127** (1997), no. 3, 417–479.

[Dia97] Fred Diamond, *The Taylor-Wiles construction and multiplicity one*, Invent. Math. **128** (1997), no. 2, 379–391.

[Gee11] Toby Gee, *Automorphic lifts of prescribed types*, Math. Ann. **350** (2011), no. 1, 107–144.

[GM92] F. Gouvêa and B. Mazur, *Families of modular eigenforms*, Math. Comp. **58** (1992), no. 198, 793–805.

[Gou01] Fernando Q. Gouvêa, *Where the slopes are*, J. Ramanujan Math. Soc. **16** (2001), no. 1, 75–99.

[HLTT13] Michael Harris, Kai-Wen Lan, Richard Taylor, and Jack Thorne, *On the rigid cohomology of certain Shimura varieties*, preprint (2013).

[HSBT10] Michael Harris, Nick Shepherd-Barron, and Richard Taylor, *A family of Calabi-Yau varieties and potential automorphy*, Ann. of Math. (2) **171** (2010), no. 2, 779–813.

[HT13] Yongquan Hu and Fucheng Tan, *The Breuil–Mézard conjecture for non-scalar split residual representations*, arXiv preprint arXiv:1309.1658 (2013).

[Kil08] L. J. P. Kilford, *On the slopes of the U_5 operator acting on overconvergent modular forms*, J. Théor. Nombres Bordeaux **20** (2008), no. 1, 165–182.

[Kis09] Mark Kisin, *The Fontaine-Mazur conjecture for* GL_2, J. Amer. Math. Soc. **22** (2009), no. 3, 641–690.

[KM12] L. J. P. Kilford and Ken McMurdy, *Slopes of the U_7 operator acting on a space of overconvergent modular forms*, LMS J. Comput. Math. **15** (2012), 113–139.

[KW09] Chandrashekhar Khare and Jean-Pierre Wintenberger, *On Serre's conjecture for 2-dimensional mod p representations of* $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Ann. of Math. (2) **169** (2009), no. 1, 229–253.

[Loe08] David Loeffler, *Explicit calculations of automorphic forms for definite unitary groups*, LMS J. Comput. Math. **11** (2008), 326–342.

[LWX14] Ruochuan Liu, Daqing Wan, and Liang Xiao, *Eigencurve over the boundary of the weight space*, 2014.

[Nak14] Kentaro Nakamura, *Zariski density of crystalline representations for any p-adic field*, J. Math. Sci. Univ. Tokyo **21** (2014), no. 1, 79–127.

[Paš15] Vytautas Paškūnas, *On the Breuil–Mézard conjecture*, Duke Math. J. **164** (2015), no. 2, 297–359.

[Roe14] David Roe, *The 3-adic eigencurve at the boundary of weight space*, Int. J. Number Theory **10** (2014), no. 7, 1791–1806.

[Sai97] Takeshi Saito, *Modular forms and p-adic Hodge theory*, Invent. Math. **129** (1997), no. 3, 607–620.

[Shi12] Sug Woo Shin, *Automorphic Plancherel density theorem*, Israel J. Math. **192** (2012), no. 1, 83–120.

[ST12] Sug-Woo Shin and Nicolas Templier, *Sato–Tate theorem for families and low-lying zeros of automorphic l-functions*, arXiv preprint arXiv:1208.1945 (2012).

[Tay08] Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations. II*, Publ. Math. Inst. Hautes Études Sci. (2008), no. 108, 183–239.

[Var14] Ila Varma, *Local-global compatibility for Galois representations associated to regular algebraic cuspidal automorphic representations when $l = p$ (in preparation)*, 2014.

[Wan98] Daqing Wan, *Dimension variation of classical and p-adic modular forms*, Invent. Math. **133** (1998), no. 2, 449–463.

[WXZ14] Daqing Wan, Liang Xiao, and Jun Zhang, *Slopes of eigencurves over boundary disks*, 2014.

E-mail address: `k.buzzard@imperial.ac.uk`

Department of Mathematics, Imperial College London

E-mail address: `toby.gee@imperial.ac.uk`

Department of Mathematics, Imperial College London