What does a general unitary group look like?

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Abstract

Just some random calculations on the general unitary group.

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1 Definition.

Let E/F be a (separable) quadratic extension of fields of characteristic not equal to 2, and we'll use a bar to denote conjugation. To define a unitary group we need a non-degenerate Hermitian sesquilinear form; this will be given by an n by n matrix $J \in \mathrm{GL}_n(E)$ with $\overline{J} = J^t$. The R-points of the general unitary group $GU(J)$ are then

$$GU(J)(R) = \{g \in \mathrm{GL}_n(E \otimes_F R) : gJ\overline{g}^t = \lambda J\}.$$

Here $\lambda \in E \otimes_F R$ if you like, but taking conjugate-transpose of everything we see instantly that $\overline{\lambda} = \lambda$ and hence we may as well assume $\lambda \in R$.

If R is in fact an E-algebra then $E \otimes_F R = R \oplus R$, conjugation becomes "switch the factors", and

$$GU(J)(R) = \{(g,h) \in \mathrm{GL}_n(R \oplus R) : (g,h)(J,J^t)(h^t,g^t) = \lambda(J,J^t)\}$$

and the resulting two equations in $\mathrm{GL}_n(R)$ are equivalent (one is the transpose of the other), and equivalent to $h = \lambda J^t g^{-t} J^{-t}$. Hence $GU(J)(R) = \mathrm{GL}_n(R) \times R^\times$, the isomorphism sending (g,h) to (g,λ) and the one the other way sending (g,λ) to $(g, \lambda J^t g^{-t} J^{-t})$.

The based root data of this group can be computed over E. Here the group becomes $\mathrm{GL}_n \times \mathrm{GL}_1$ and so we have the usual story: we can use the upper triangular matrices in GL_n and so on, and get the usual $X^* = X^*(T) \oplus \mathbf{Z}$ and so on (T the torus in GL_n).

Now we need to compute μ, the action of the Galois group on this gadget. Well, let's use $\mathrm{GL}_n \times \mathrm{GL}_1$ coordinates. Imagine we start with an element (b,λ) of the standard Borel. Hitting this with Galois gives the element..., let's move to the other coordinates. This sends (b,λ) to $(b, \lambda J^t b^{-t} J^{-t})$. We now have to work out what Galois does to this. This is a bit subtle. Galois acts non-linearly and is different to the "switching" we saw earlier: the "switching", which we used a bar for, was all coming from the Galois action on the E in $E \otimes_F R$. We are now considering $R = \overline{F}$ and Galois is acting on the right. Let's let c denote this. Before we had $\overline{e \otimes r} = \overline{e} \otimes r$. If $R = E$ then we have $E \otimes R = E \oplus E$ via the map $e \otimes r \mapsto (er, \overline{e}r)$. Via this identification we see that $e \otimes \overline{r}$ becomes $(e\overline{r}, \overline{er})$ which is "switch and hit with Galois", so the Galois action on $GU(J)(R)$ when $R = E$ sends (using our $\mathrm{GL}_n \times GL_1$ coordinates) (g,λ) to $(\lambda J \overline{g}^{-t} J^{-1}, \mu)$ and we can work out μ by translating back into unitary group coordinates, where we see $\lambda = \mu$. So the Galois action on $\mathrm{GL}_n(E) \times \mathrm{GL}_1(E)$, regarded as the E-points of a variety over F, sends (g,λ) to $(\lambda J \overline{g}^{-t} J^{-1}, \lambda)$, and lo and behold the fixed points are precisely the (g,λ) with $gJ\overline{g}^t = \lambda J$, which is just what we expected now we've warmed up.

Right, so how do we compute μ_G? It suffices to compute $\mu_G(c)$. We use the $\mathrm{GL}_n \times \mathrm{GL}_1$ model. Here's how it goes. We start with (b,λ) in the Borel. We hit with c and get $(\lambda J \overline{b}^{-t} J^{-1}, \lambda)$. We conjugate until we're back in B, so we may as well go to $(\lambda \Phi \overline{b}^{-t} \Phi^{-1}, \lambda)$, with Φ the antidiagonal matrix with alternating $+1$s and -1s up the antidiagonal. So now we can see how Galois is acting on $\mathrm{GL}_n \times \mathrm{GL}_1$: it's induced by the map sending (b,λ) to $(\lambda \Phi b^{-t} \Phi^{-1}, \lambda)$.

2 The case $n = 2$.

In this case the character sending $(\mathrm{diag}(\mu,\nu),\lambda)$ to $\mu^a\nu^b\lambda^c$ gets sent, by Galois, to the character sending it to $(\lambda\nu^{-1})^a(\lambda\mu^{-1})^b\lambda^c$, which is $\mu^{-b}\nu^{-a}\lambda^{a+b+c}$. So Galois is represented (on X^*) by the matrix

$$c := \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}.$$

Note that the root, corresponding to $a = 1$, $b = -1$, $c = 0$, remains fixed.

It's easy to check that Galois is represented on the dual based root datum by the transpose of this matrix, namely

$$c^t = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Now I claim that these based root data together with their Galois actions are not isomorphic. Here is a very short proof. Any isomorphism of based root data had better send the positive root to the positive root! The positive root in both cases is $(1,-1,0)^t$ and note that in both cases this is fixed by Galois. Because the trace of c is 1 and $c^2 = 1$, both c and c^t have a 1-dimensional eigenspace for the eigenvalue -1. For c this eigenspace is spanned by $(1,1,-1)^t$ and for c^t it's spanned by $(1,1,0)^t$. Any isomorphism had better hence send one of these vectors to plus or minus the other. But modulo 2 one of these vectors is congruent to the roots, and the other one isn't, and the isomorphism sends a root to a root, so it can't exist.

Conclusion: the based root datum for $GU(2)$, with its Galois action, is not self-dual.

3 L-groups.

Back to the general case now. If we think of an element of the torus T (over the algebraic closure) as $(\mu_1,\mu_2,\ldots,\mu_n)$ then the action of c on the torus of the general unitary group sends (with our $GL_n \times \mathrm{GL}_1$ coordinates as usual) $((\mu_1,\ldots,\mu_n),\lambda)$ to $\lambda(\mu_n^{-1},\ldots,\mu_1^{-1}),\lambda)$. So the matrix representing c on X^* is just the obvious generalisation of the c above (an anti-diagonal of -1s, on top of a row of 1s), and so c on the dual based root datum is an anti-diagonal of -1s next to a colum of 1s. We need to translate this into an action of c on $\mathrm{GL}_n \times \mathrm{GL}_1$, and one checks that it's this: $c(g,\lambda) = (\Phi g^{-t}\Phi^{-1}, \det(g)\lambda)$.

I want to assert that for $n = 2$ this group is not isomorphic to the CHT group, namely $\mathrm{GL}_2 \times \mathrm{GL}_1$ with $c(g,\lambda) = (\lambda g^{-t},\lambda)$.

So let's say there there's an isomorphism taking one c into the other. Note that it's easy to understand all maps $\mathrm{GL}_2\times\mathrm{GL}_1 \to \mathrm{GL}_2\times\mathrm{GL}_1$ because each such is the sum of a 2-dimensional and a 1-dimensional representation of $\mathrm{GL}_2\times\mathrm{GL}_1$, and we can list these using standard facts about representation theory of reductive groups. Note also that inverse-transpose doesn't come into it, because $g \in \mathrm{GL}_2$ is conjugate to a twist of its inverse-transpose.

Next we have to understand all the possibilities for complex conjugation on the L-group. Let's list the elements of order 2 in the non-identity component of the L-group. They're of the form $(Y,\mu)c$ with $(Y,\mu)(\Phi Y^{-t}\Phi^{-1}, \det(Y)\mu) = 1$, so we have $Y\Phi Y^{-t} = \Phi$ and $\mu^2\det(Y) = 1$. Well, such things are going to exist in general. Let's fix one, and let's let C denote the corresponding automorphism of $\mathrm{GL}_n \times \mathrm{GL}_1$. Explicitly, we have

$$\begin{aligned} C(g,\lambda) &= (Y,\mu)c(gY,\lambda\mu) = (Y,\mu)(\Phi g^{-t}Y^{-t}\Phi^{-1}, \det(g)\det(Y)\lambda\mu) \\ &= (Y\Phi g^{-t}Y^{-t}\Phi^{-1}, \det(g)\lambda) = (Y\Phi g^{-t}\Phi^{-1}Y^{-1}, \det(g)\lambda) \end{aligned}$$

. Hmm, so it's just come out as conjugation anyway…,actually, that was always going to happen wasn't it.

Let's say the isomorphism from the CHT group to the L-group sends (g,λ) to

$$i(g,\lambda) := (XgX^{-1}\det(g)^a\lambda^b, \det(g)^c\lambda^d)$$

for some integers a, b, c, d. Applying the isomorphism i and then the L-group's c, starting with (g, λ), we get $C(XgX^{-1}\det(g)^a\lambda^b, \det(g)^c\lambda^d)$, which is

$$(Y\Phi X^{-t}g^{-t}X^t\det(g)^{-a}\lambda^{-b}\Phi^{-1}Y^{-1}, \det(g)^{1+2a+c}\lambda^{2b+d}).$$

Applying on the other hand c and then the isomorphism gives us the isomorphism applied to $(\lambda g^{-t}, \lambda)$, which is

$$(X\lambda g^{-t}X^{-1}\lambda^{2a+b}\det(g)^{-a}, \lambda^{2c+d}\det(g)^{-c}).$$

Now these two last displayed things can't be equal, because look at the $\det(g)$ factor in the GL_1 component.

Conclusion: the CHT group really is not the L-group of the general unitary group when $n = 2$.