Weight one Eisenstein series.

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This is a summary of Miyake's write-up on weight one Eisenstein series. Let χ be an even Dirichlet character of conductor L and let ψ be an odd Dirichlet character of conductor M. If $L = 1$ then let c_0 be $L(\psi, 0)/2$ (this is related to a Generalized Bernoulli Number, see below) and let c_0 be zero otherwise. For $n \geq 1$ let c_n denote $\sum_{0<d|n} \psi(d)\chi(n/d)$. Then

$$\sum_{n\geq 0} c_n q^n$$

is a new Eisenstein series of level LM and character $\chi\psi$, and all of them arise in this way. I wrote a magma script to compute these things.

The special value of the L-function, explicitly, is as follows. Let ψ be a primitive odd Dirichlet character of conductor M. Then $L(\psi, 0)$ is morally some attempt to sum $\psi(n)$ for all $n \geq 1$. Well, the sum up to $n = M$ is zero because ψ can't be trivial as it's odd. So the infinite sum takes on a finite number of values, each $1/M$ times, as it were, and so one guess as to what it should be is the average of these values. So we suspect

$$\begin{aligned}
L(\psi, 0) &= 1/M \sum_{i=0}^{M-1} \sum_{j=0}^{i} \psi(j) \\
&= (1/M) \sum_{0\leq j\leq i\leq M-1} \psi(j) \\
&= (1/M) \sum_{j=0}^{M-1} (M - j)\psi(j) \\
&= (-1/M) \sum_{j=0}^{M-1} j\psi(j)
\end{aligned}$$

which makes the Eisenstein series in this case equal to

$$(-1/2M)\sum_{j=0}^{M-1} j\psi(j) + \sum_{n\geq 1}\left(\sum_{0<d|n}\psi(d)\right)q^n.$$

For example, if $M = 3$ and ψ is non-trivial and χ is trivial we see that $(-1/6)(\psi(1)+2\psi(2))+\sum_{n\geq 1}(\sum_{d|n}\psi(d))q^n$ is an Eisenstein serires, and the constant term is $1/6$ so we get that $1+6q+6q^3+\ldots$ is an Eisenstein series, and this is just $\sum_{a,b} q^{a^2+ab+b^2}$, the theta series.